HIERARCHICAL REFINEMENTS OF BILINEAR FINITE ELEMENTS

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Abstract

We show some of the properties of the algebraic multilevel iterative methods when the hierarchical bases of finite elements with rectangular supports are used. In particular, we study two types of hierarchy, the so called h- and p-hierarchical finite element spaces.

1 Hierarchical Bases of Finite Elements

We assume a weak formulation of a partial differential elliptic equation,

$$a(u,v) = (f,v), \tag{1}$$

for $u, v \in H_0^1(\Omega)$, Ω polygonal domain in R^2,

$$a(u,v) = \int_\Omega A\nabla u.\nabla v \,\mathrm{d}x \qquad \text{and} \qquad (f,v) = \int_\Omega fv \,\mathrm{d}x, \tag{2}$$

A symmetric positive definite in Ω.

We suppose a FE approximation V of $H_0^1(\Omega)$. The algebraic multilevel iterative (AMLI) method in its two-level form exploits a splitting of the FE space into two hierarchical spaces of functions, let us denote the coarse grid space by U and the space of functions corresponding to the added nodes of the fine grid by W, and let

$$V = U \oplus W.$$

Then the equation (1) can be transformed into a 2×2 block form

$$\begin{pmatrix} B_W & B_{WU} \\ B_{UW} & B_U \end{pmatrix} \begin{pmatrix} \tilde{u}_W \\ \tilde{u}_U \end{pmatrix} = \begin{pmatrix} F_W \\ F_U \end{pmatrix}, \tag{3}$$

where $\tilde{u}_W$ and $\tilde{u}_U$ are the coefficients of the solution with respect to the basis of U and of W, respectively.

We can choose the block diagonal of the matrix B of the system (3) as for the preconditioning matrix M_{add} (additive form) or we can use the block Gauss-Seidel preconditioning M_{mult} (multiplicative form). Thus the preconditioned system (3) has the condition number bounded by

$$\kappa(M_{add}^{-1}B) \leq \frac{1+\gamma}{1-\gamma}$$

and

$$\kappa(M_{mult}^{-1}B) \leq \frac{1}{1-\gamma^2},$$

respectively, where γ is the constant in the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality

$$|a(u,w)| \leq \gamma\sqrt{a(u,u)a(w,w)},$$

$u \in U$, $w \in W$. See e.g. [2, 3] and the references therein for more detailed theory.

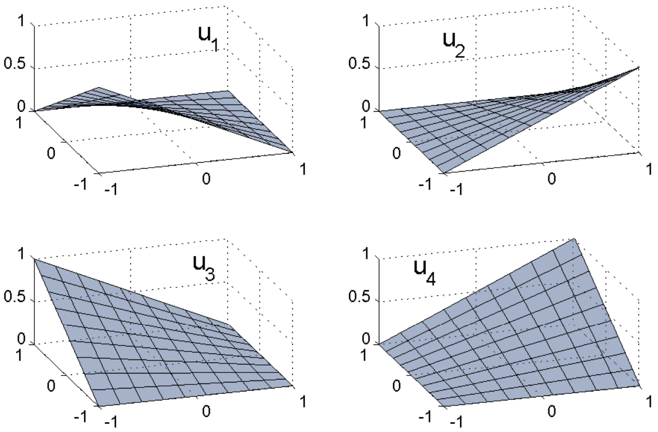

Figure 1: Four coarse grid functions of U on a reference macroelement.

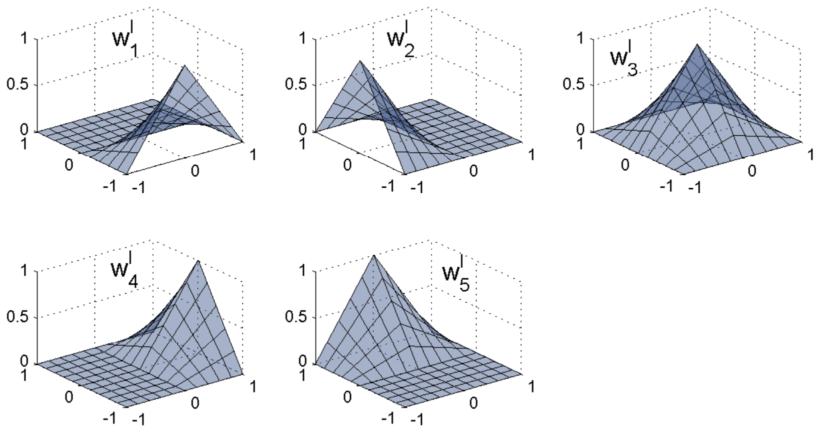

Figure 2: Five fine grid functions of W_h corresponding to h-refinement on a reference macroelement.

2 CBS Constants for h- and p-hierarchy on rectangular elements

In this paper we consider two different refinements of the space of bilinear FEs on rectangles. The coarse space U contains piecewise bilinear functions, i.e. on one macroelement, there are four bilinear functions, see Figure 1. The fine grid space W_h corresponding to the h-refinement of U is represented by five piecewise bilinear functions on a reference macroelement, see Figure 2, while the space W_p corresponding to the p-refinement of U is represented by five functions of the polynomial space Q_2 on each macroelement, which can be seen on Figure 3.

We compute the uniform estimates of the CBS constants for these two refinements of bilinear FEs. Obviously, the upper bound for γ less than one indicates that the corresponding splitting can be used for the AMLI methods. Several hierarchical FE spaces have been studied and the uniform CBS constant estimates were found, see e.g. [2, 3, 4, 5, 7]. In this paper we introduce some new results. The first result is that the CBS constant is uniformly bounded in case of p-hierarchy of bilinear FEs when matrix A in equation (2) is diagonal. The upper bound is slightly greater than that for h-refinement. In comparison, the p-refinement of the linear FEs on triangles yields the uniform CBS constant estimate equal to one, thus this splitting is unusable for AMLI methods [5].

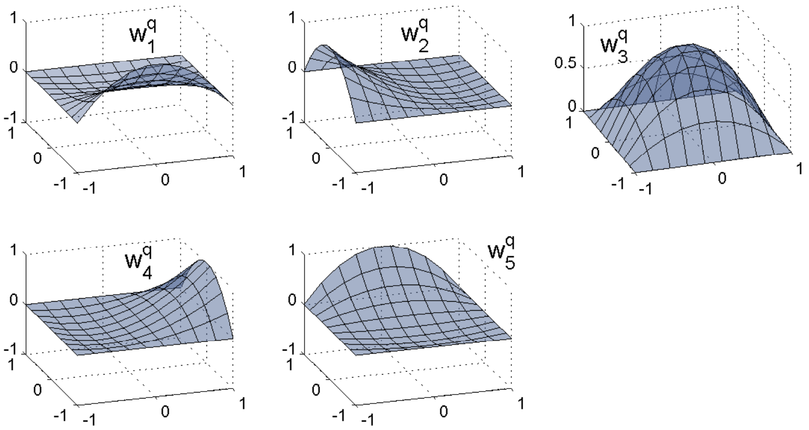

Figure 3: Five fine grid functions of W_p corresponding to p-refinement on a reference macroelement.

Theorem 1. *The CBS constants for the hierarchical h-refinement of bilinear FEs and for the equation (1) with a diagonal matrix A in (2) are not greater than*

$$\sqrt{\frac{3}{8}} \qquad \textit{and} \qquad \sqrt{\frac{3}{4}},$$

respectively, for isotropic operator and regular elements and for anisotropic equation or elements, respectively. The CBS constants for the hierarchical p-refinement of bilinear FEs are not greater than

$$\sqrt{\frac{5}{11}} \qquad \textit{and} \qquad \sqrt{\frac{9}{11}},$$

respectively, for isotropic operator and regular elements and for anisotropic equation or elements, respectively.

In the case when matrix A in (2) is positive definite but not diagonal, the uniform CBS constatnt estimate is equal to one. Thus for such equations the AMLI methods don't yield better convergence than one-level iterative solvers.

3 Robust Preconditioning

In papers [1, 3] the new idea of constructing a preconditioning matrix C_W for the block B_W which leads to the uniformly bounded condition number has been introduced. The hierarchical linear FEs on triangles are considered there.

The preconditioning matrix C_W is assembled element by element from the macroelement stiffness matrices which correspond to the space W. From these particular matrices, some of the off-diagonal entries are deleted in such manner that after a proper reordering the basis functions of W (reordering rows and columns in C_W), the preconditioning matrix C_W is tri-diagonal. The off-diagonal elements which are not deleted from the macroelement stiffness matrices can be called "strong connections" for the purpose of this paper.

We show that this idea can be adopted for the hierarchical h- and p-refinement of bilinear FEs as well. The strong connections are graphically displayed on Figure 4 for five basis functions of W on a macroelement and for h-refinement of bilinear FEs. On Figure 5, the strong connections for p-refinement of bilinear FEs are marked. The connections marked by α (solid lines) are kept when

$$\frac{A_{11}}{A_{22}} \geq \frac{d_1^2}{d_2^2}$$

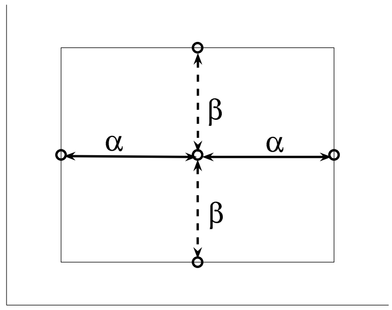

Figure 4: "Strong connections" among functions of W_h on a macroelement for h-refinement of bilinear FEs.

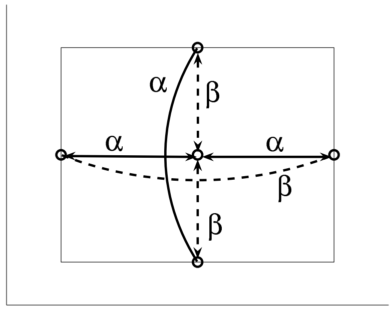

Figure 5: "Strong connections" among functions of W_p on a macroelement for p-refinement of bilinear FEs.

where A_{11} and A_{22} are the diagonal elements of matrix A in (2) and $d_1 \times d_2$ is the size of the particular coarse element, otherwise the entries corresponding to the connections β (dashed lines) are presented in the preconditionig matrix.

The examples of numbering the basis functions of the refining space W for case of h- and p-refinement are presented in Figure 6 and in Figure 7, respectively. We consider six coarse elements in these examples.

Theorem 2. *There exists generalized tri-diagonal preconditioning matrices such that the preconditioned diagonal blocks B_W have the condition numbers not greater that* 4.2 *and* 8.2, *respectively, for the hierarchical h- and p-refinement of bilinear FEs, respectively.*

In paper [6] more details on this issue are provided.

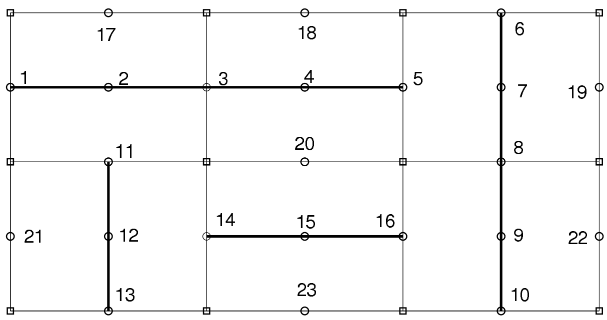

Figure 6: An example of reordering the elements of W_h in h-refined bilinear FEs.

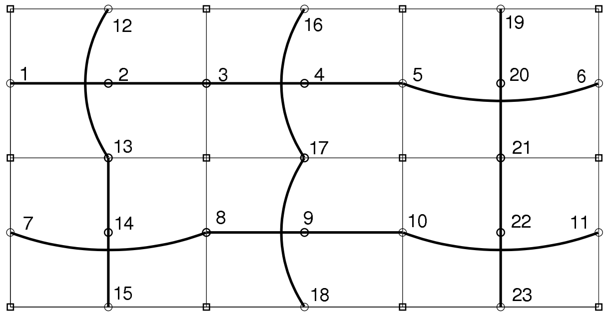

Figure 7: An example of reordering the elements of W_p in p-refined bilinear FEs.

All of the estimates presented in this paper have been obtained after the singular value decomposition and the eigenvalue computation for the concerned matrices which were first numerically performed with help of Matlab software and then proved analytically.

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