USE OF MATLAB FOR DOMAIN DECOMPOSITION METHOD FOR CONTACT PROBLEM IN ELASTICITY

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Abstract

In the present paper we will deal with numerical solution of a generalized semi-coercive contact problem in linear elasticity, for the case that several bodies of arbitrary shapes are in mutual contacts and are loaded by external forces, by using the non-overlapping domain decomposition and finite elements method. The numerical example will be presented.

1 Introduction

In mechanics as well as technological practice there are problems whose investigations lead to solving model problems based on variational formulations. Such problems are described frequently by variational inequalities. Variational inequalities physically describe the principle of virtual work in its inequality form. The numerical solution is based on the theory of contact problem in elasticity and the finite element approximation. The algorithm used for our computation is based on the nonoverlapping domain decomposition method. For an extensive problems it is appropriate to use programming language as `FORTRAN` or `C++`. System `MATLAB` play an important role in testing, debugging and last but not least in vizualization.

2 The model

We consider a system of elasic bodies decomposed into subdomains each of which occupies, in reference configuration, a domain Ω^n in $\mathbb{R}^2$, $n = 1, \dots, N$, with sufficiently smooth boundary $\partial\Omega^n$. Suppose that boundary $\bigcup_{n=1}^{N} \partial\Omega^n$ consists of disjoint parts Γ_u, Γ_o, Γ_τ, Γ_c and Γ. By Γ_u we denote the part of boundary on that displacements are prescribed. The part Γ_c denotes the part of boundary that may get into unilateral contact with some other subdomain, the part Γ_o denote the part of boundary on that is prescribed the condition of the bilateral contact, the part Γ_τ denotes the loaded part of boundary and the part Γ denotes inner interface between subdomains. We suppose that $\overline{\Gamma} \cap \overline{\Gamma}_\tau = \emptyset$. Let body forces $\mathbf{F}$, surface traction $\mathbf{P}$ and displacements $\mathbf{u}_0$ be given (see Fig. 1).

We shall look for the displacement that satisfy the conditions of equilibrium in the set $K = \{\mathbf{v} \in V |\, v_{kn} + v_{ln} \leq 0 \text{ on } \Gamma_c\}$ of all kinematically admissible displacement $\mathbf{v} \in V$, $V = \{\mathbf{v} \in \mathcal{H}^1(\Omega) |\ \mathbf{v} = \mathbf{u}_0 \text{ on } \Gamma_u,\, v_n = 0 \text{ on } \Gamma_o\}$, $\mathcal{H}^1(\Omega) = [H^1(\Omega^1)]^2 \times \cdots \times [H^1(\Omega^N)]^2$ is standard Sobolev space. The displacement $\mathbf{u} \in K$ of the system of bodies in equilibrium then minimizes the energy functional $\mathcal{L}(\mathbf{v}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v})$:

$$\mathcal{L}(\mathbf{u}) \leq \mathcal{L}(\mathbf{v}) \text{ for any } \mathbf{v} \in K, \tag{1}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \sum_{n=1}^{N} \int_{\Omega^n} c_{ijkl}^n e_{ij}\left(\mathbf{u}^n\right) e_{kl}\left(\mathbf{v}^n\right) d\mathbf{x}, \tag{2}$$

$$L(\mathbf{v}) = \sum_{n=1}^{N} \int_{\Omega^n} F_i^n d\mathbf{x} + \sum_{n=1}^{N} \int_{\Gamma_\tau \cap \partial\Omega^n} P_i^n v_i^n \, ds. \tag{3}$$

3 Domain decomposition algorithm

Let us introduce

$$T = \{n \in \{1, \ldots, N\} : \ \bar{\Gamma}_c \cap \partial\bar{\Omega}^n = \emptyset\}$$

the set of all indices of subdomains Ω^n which are not adjacent to a contact, and

$$\vartheta = \{[k, l], \ k, l \in \{1, \ldots, N\} : \partial\bar{\Omega}^k \cap \partial\bar{\Omega}^l \subset \Gamma_c\}$$

represents couples of subdomains in unilateral contact. Suppose that $\Gamma \cap \Gamma_c = \emptyset$. Then for the

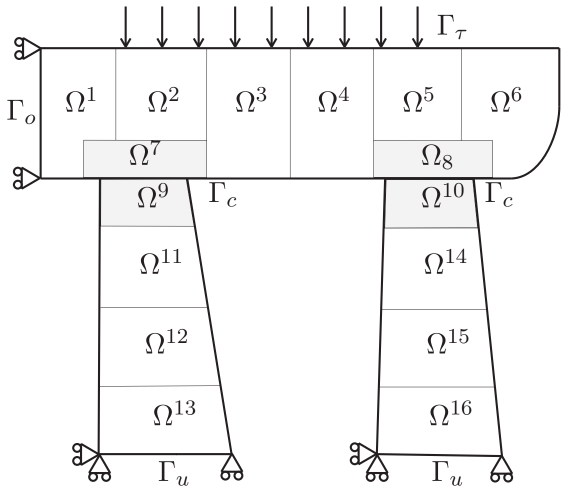

Figure 1: The contact problem with decomposition

trace operator $\gamma : [H^1(\Omega^n)]^2 \to [L^2(\partial\Omega^n)]^2$ we have

$$V_\Gamma = \gamma K|_\Gamma = \gamma V|_\Gamma. \tag{4}$$

Let $\gamma^{-1} : V_\Gamma \in V$ be an arbitrary linear inverse mapping satisfying

$$\gamma^{-1}\overline{\mathbf{v}} \ = 0 \quad \text{on } \Gamma_c \quad \forall \overline{\mathbf{v}} \in V_\Gamma. \tag{5}$$

Let us introduce restrictions $\overline{R}_n : V_\Gamma \to \Gamma_n$; $L_n : L \to \Omega^n$; $a_n(.,.) : a(.,.) \to \Omega^n$; $V(\Omega^n) : V \to \Omega^n$ and let

$$V^0(\Omega^n) = \{\mathbf{v} \in V \mid \mathbf{v} = \mathbf{0} \ \text{ on } \ \overline{(\cup_{n=1}^N \Omega^n) \backslash \Omega^n}\}$$

be the space of functions with zero traces on Γ_n where $\Gamma_n = \Gamma \cap \partial\Omega^n$ The algorithm is based on the next theorem and on the use of local and global Schur complements.

Theorem: A function $\mathbf{u}$ is a solution of a global problem (1), if and only if its trace $\overline{\mathbf{u}} = \gamma\mathbf{u}|_\Gamma$ on the interface Γ satisfies the condition

$$\sum_{i=1}^{N} [a_i(\mathbf{u}_i(\overline{\mathbf{u}}), \gamma^{-1}\overline{\mathbf{w}}) - L_i(\gamma^{-1}\overline{\mathbf{w}})] = 0, \qquad \forall \overline{\mathbf{w}} \in V_\Gamma, \overline{\mathbf{u}} \in V_\Gamma \tag{6}$$

and its restrictions $\mathbf{u}_i(\mathbf{u}) \equiv \mathbf{u}|_{\Omega^i}$ satisfy:

(i) the condition

$$a_i(\mathbf{u}_i(\overline{\mathbf{u}}), \varphi_i) = L_i(\varphi_i), \quad \forall \varphi_i \in V^0(\Omega^i), \quad \text{for } i \in T, \tag{7}$$

(ii) the condition

$$a_k(\mathbf{u}_k(\overline{\mathbf{u}}), \varphi_k) + a_l(\mathbf{u}_l(\overline{\mathbf{u}}), \varphi_l) \geq L_k(\varphi_k) + L_l(\varphi_l), \ \forall \varphi_i \in V^0(\Omega^i), \ i = k, l, \quad \text{for } [k, l] \in \vartheta. \tag{8}$$

Proof. See [4].

To analyze the condition (6) the **local and global Schur complements** are introduced. Let

$$V_i = \{\gamma \mathbf{v}_{|\Gamma_i} \mid \mathbf{v} \in K\} = \{\gamma \mathbf{v}_{|\Gamma_i} \mid \mathbf{v} \in V\}$$

and define a particular case of the restriction of the inverse mapping $\gamma^{-1}(.)|_{\Omega^i}$ by

$$\begin{cases} Tr_i^{-1} : V_i \to V(\Omega^i), \quad \gamma(Tr_i^{-1}\overline{\mathbf{u}})|_{\Gamma_i} = \mathbf{u}_i, \ i = 1, \dots, N, \\ a_i(Tr_i^{-1}\overline{\mathbf{u}}_i, \mathbf{v}_i) = 0, \quad \forall \mathbf{v}_i \in V^0(\Omega^i), \ Tr_i^{-1}\overline{\mathbf{u}}_i \in V(\Omega^i), \text{ for } i \in T. \end{cases} \tag{9}$$

For $[k,l] \in \vartheta$ we complete the definition by the boundary condition (5), i.e.

$$Tr_k^{-1}\overline{\mathbf{u}}_k + Tr_l^{-1}\overline{\mathbf{u}}_l = 0 \text{ on } \Gamma_c. \tag{10}$$

The local Schur complement for $i \in T$ is the operator $\mathcal{S}_i : V_i \to (V_i)^*$ defined by

$$\langle \mathcal{S}_i \overline{\mathbf{u}}_i, \overline{\mathbf{v}}_i \rangle = a_i(Tr_i^{-1}\overline{\mathbf{u}}_i, Tr_i^{-1}\overline{\mathbf{v}}_i) \quad \forall \overline{\mathbf{u}}_i, \overline{\mathbf{v}}_i \in V_i. \tag{11}$$

For subdomains which are in contact we define a **common local Schur complement** for the union $\Omega^k \cup \Omega^l$ (where $[k,l] \in \vartheta$) as the operator $\mathcal{S}_{k,l} : (V_k \times V_l) \to (V_k \times V_l)^* = (V_k)^* \times (V_l)^*$ defined by

$$\langle \mathcal{S}_{k,l}(\overline{\mathbf{y}}_k, \overline{\mathbf{y}}_l), (\overline{\mathbf{v}}_k, \overline{\mathbf{v}}_l) \rangle = a_k(\mathbf{u}_k(\overline{\mathbf{y}}_k), Tr_k^{-1}\overline{\mathbf{v}}_k) + a_l(\mathbf{u}_l(\overline{\mathbf{y}}_l), Tr_l^{-1}\overline{\mathbf{v}}_l) \qquad \forall (\overline{\mathbf{v}}_k, \overline{\mathbf{v}}_l) \in V_k \times V_l, \tag{12}$$

where Tr_k^{-1} and Tr_l^{-1} are defined by means of (9) and (10).

The condition (6) can be expressed by means of local Schur complements in the form

$$\sum_{i \in T} \langle \mathcal{S}_i \overline{\mathbf{u}}_i, \overline{\mathbf{v}}_i \rangle + \sum_{[k,l] \in \vartheta} \langle \mathcal{S}_{k,l}(\overline{\mathbf{u}}_k, \overline{\mathbf{u}}_l), (\overline{\mathbf{v}}_k, \overline{\mathbf{v}}_l) \rangle = \sum_{i=1}^{N} L_i(Tr_i^{-1}\overline{\mathbf{v}}_i) \quad \forall \overline{\mathbf{v}} \in V_\Gamma, \tag{13}$$

where $\overline{\mathbf{u}} = \gamma \mathbf{u}_{|\Gamma}$, $\overline{\mathbf{v}}_i = \overline{R}_i \overline{\mathbf{v}}$, $\overline{\mathbf{u}}_i = \overline{R}_i \overline{\mathbf{u}}$. Then we will solve the equation (13) on the interface Γ in the dual space $(V_\Gamma)^*$. We rewrite (13) into the following form

$$\mathcal{S}_0 \overline{\mathbf{U}} + \mathcal{S}_{CON} \overline{\mathbf{U}} = \mathcal{F}, \tag{14}$$

where

$$\begin{aligned} \mathcal{S}_0 &= \sum_{i \in T} (\overline{R}_i)^T \mathcal{S}_i \overline{R}_i, \\ \mathcal{S}_{CON} &= \sum_{[k,l] \in \vartheta} \overline{R}_{k,l}^T \mathcal{S}_{k,l} \overline{R}_{k,l}, \\ \mathcal{F} &= \sum_{i=1}^{N} (\overline{R}_i)^T (Tr_i^{-1})^T L_i \end{aligned} \tag{15}$$

and $\overline{R}_{k,l}(\overline{\mathbf{u}}) = (\overline{R}_k(\overline{\mathbf{u}}), \overline{R}_l(\overline{\mathbf{u}}))^T, \overline{\mathbf{u}} \in V_\Gamma$.

Equation (14) will be solved by **successive approximations**, because the operators $\mathcal{S}_{k,l}$ and therefore $\mathcal{S}_{CON}$ are nonlinear. As a initial approximation $\overline{\mathbf{U}}^0$ we choose the solution of the global primal problem, where the boundary conditions on Γ_c are replaced by the linear bilateral conditions

$$u_{kn} - u_{ln} = 0, \quad \text{on } \Gamma_c. \tag{16}$$

Then we replace the set K by $K^0 = \{\mathbf{v} \in V \mid v_{kn} - v_{ln} = 0 \text{ on } \Gamma_c\}$ and therefore, we solve the following problem

$$\mathbf{u}^0 = \arg\min_{\mathbf{v} \in K^0} \mathcal{L}(\mathbf{v}), \tag{17}$$

where $\mathcal{L}(\mathbf{v}) = \frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v})$ and we set $\overline{\mathbf{U}}^0 = \gamma\mathbf{u}^0|_\Gamma$. The auxiliary problem (17) represents a linear elliptic boundary value problem with bilateral contact and it can be solved by the domain decomposition method again.

The non-linear equation (14) will be solved by successive approximations. We will assume that the approximation $\overline{\mathbf{U}}^{k-1}$ is known and the next approximation $\overline{\mathbf{U}}^{k}$we find as the solution of the following linear problem

$$\mathcal{S}_0\overline{\mathbf{U}}^k = \mathcal{F} - \mathcal{S}_{CON}\overline{\mathbf{U}}^{k-1}, k = 1, 2, \ldots \tag{18}$$

In [4] the convergence of the method of successive approximation (18) to the solution of the original problem (14) in the space $(V_\Gamma)^*$ is proved.

Numerically (17) and (18) are solved by the finite element method. Let $V_h, K_h^0 = V_h \cap K^0$ be finite element approximations of the sets of virtual and admissible displacements. The finite element approximation of (17) leads to solve the following problem: find a function $\mathbf{u}_h \in K_h$, such that

$$a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) = L(\mathbf{v}_h - \mathbf{u}_h) \qquad \forall \mathbf{v}_h \in K_h^0. \tag{19}$$

4 Numerical results

The practical behavior of presented algorithm is illustrated on geomechanical model problem describing loaded granite block with cracs. A geometry of the problem is in Fig. 2.

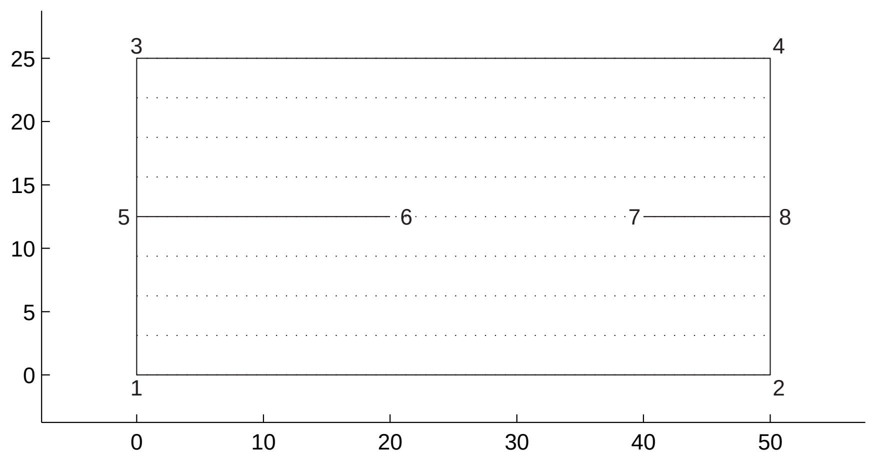

Figure 2: A geometry of the geomechanical problem

Material parameters: Young's modulus $E = 5.2 \times 10^9$[Pa] and Poisson's ratio $\nu = 0, 18$.

Boundary conditions: Prescribed zero displacement on 1-2. Pressure $0, 6 \times 10^7$[Pa] on 2-8 and 3-5 and pressure $0, 3 \times 10^7$[Pa] on 3-4. Zero pressure on 1-5 and 4-8. Unilateral contact boundary on 5-6 and 7-8. The dash lines represent the inner interface Γ.

Discretization statistics: 8 subdomains, 3888 nodes, 4800 elements, 48 unilateral contact conditions, 519 interface elements.

Figs. 3, 4 and 5 show the deformations (enlarge factor is 50) and the horizontal and the vertical components of displacement. Figs. 6, 7 and 8 represent the horizontal component τ_x, the vertical component τ_y and the shear component τ_{xy} of stress tensor. Fig. 9 demonstrates principal stresses in granite block. On this figure symbol $\leftarrow\rightarrow$ denotes extension and symbol $\rightarrow\leftarrow$ denotes compression. For figures generating, I exploited `pdetool` - the partial differential equation toolbox (for example function `pdeplot`).

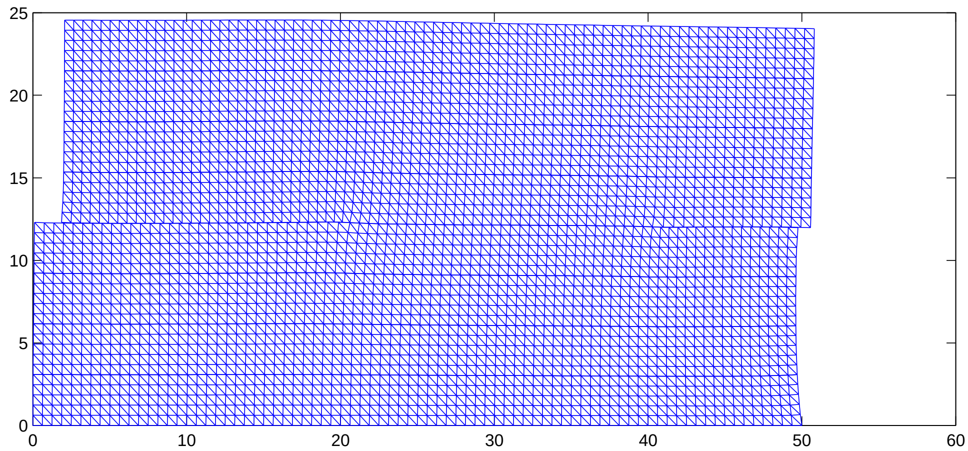

Figure 3: The deformations (enlarge factor is 50)

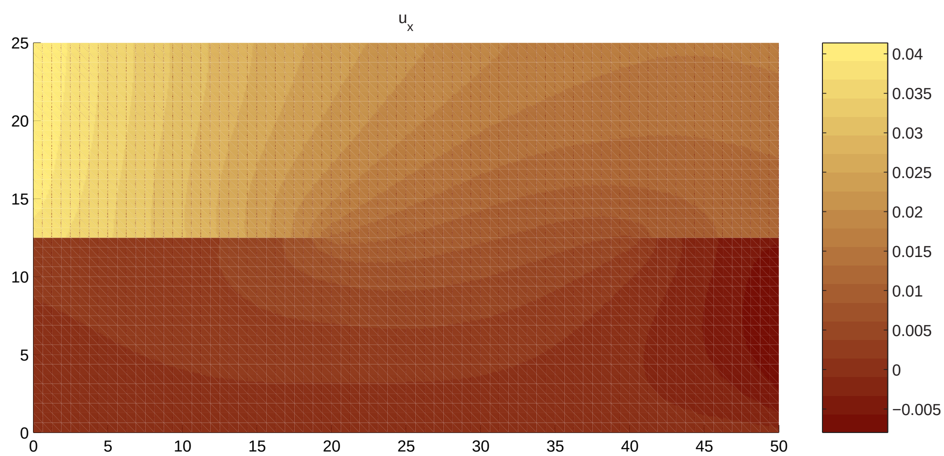

Figure 4: The horizontal component of displacement

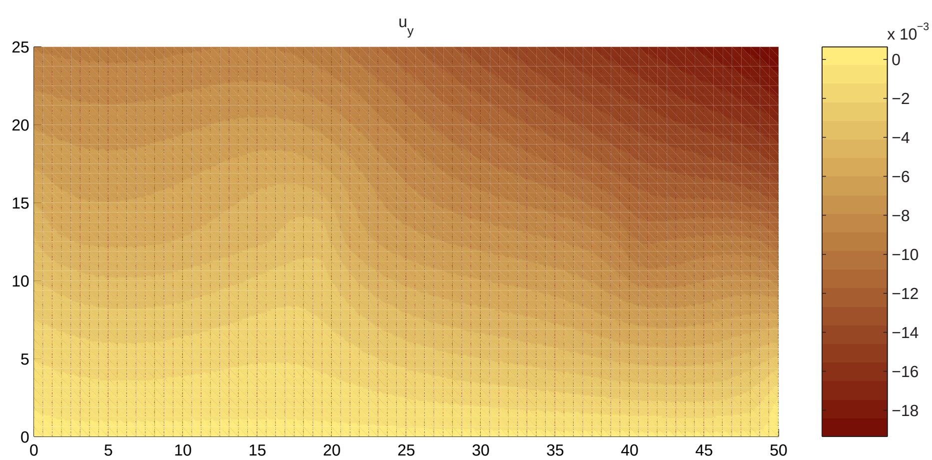

Figure 5: The vertical component of displacement

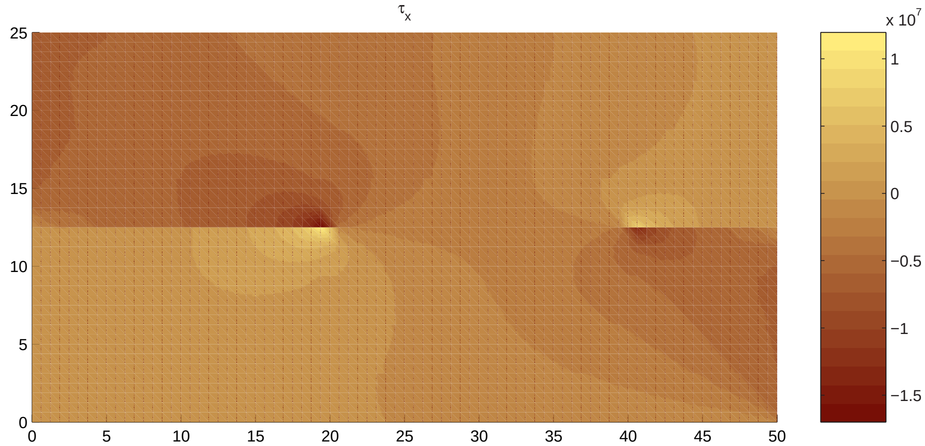

Figure 6: The horizontal component of stress tensor

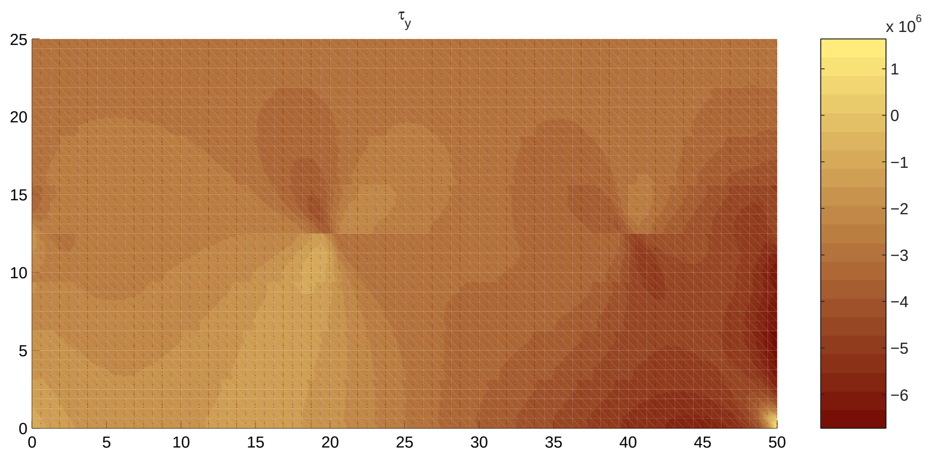

Figure 7: The vertical component of stress tensor

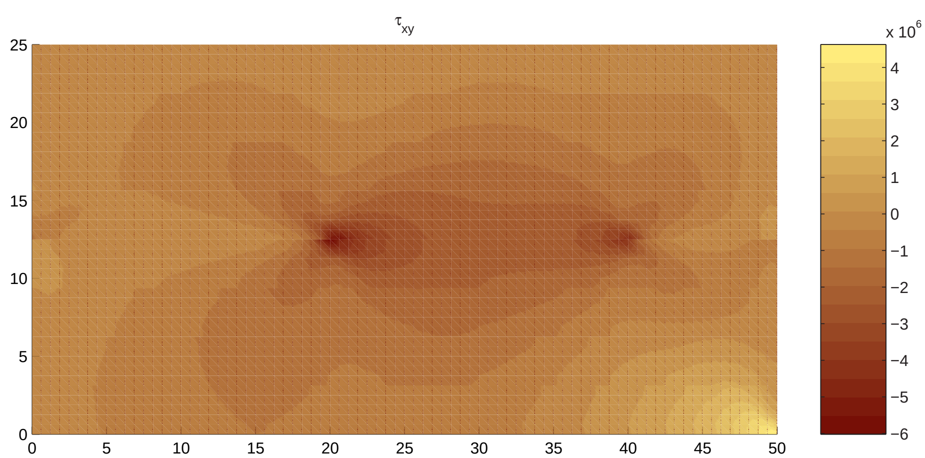

Figure 8: The shear stresses

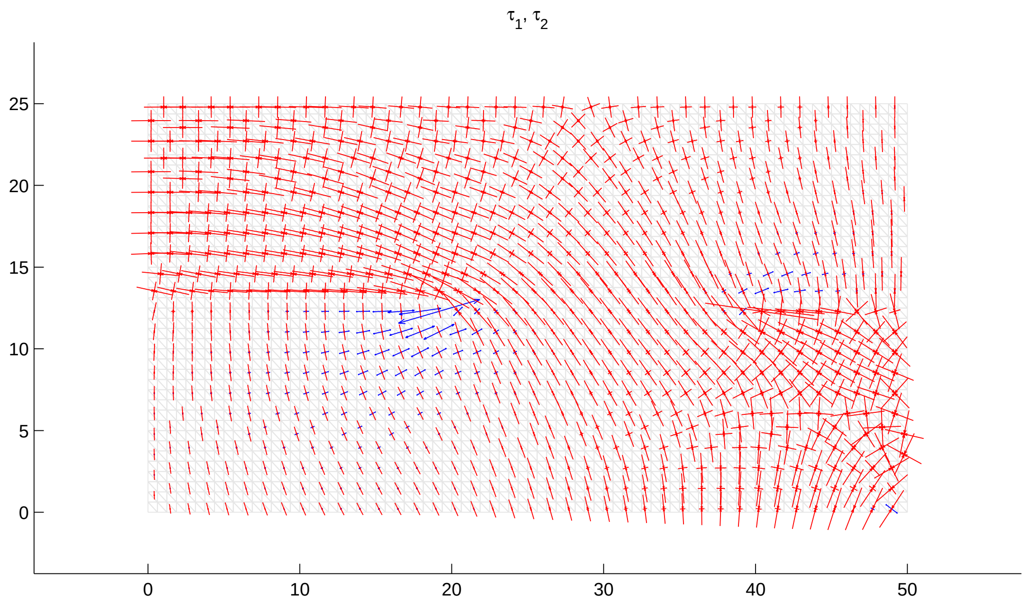

Figure 9: The principal stresses

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