WAVELET TRANSFORMS IN ITERATIVE SIGNAL REGIONS RECOVERY

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Abstract

The paper presents basic principles of the iterative use of wavelet de-noising principles for the recovery of corrupted or missing signal regions. The main part of the contribution is devoted to the study of proposed threshold methods for modification of wavelet decomposition coefficients. Suggested methods include generalization of soft and hard thresholding algorithms introducing transition period to enable the use of continuous and differentiable threshold functions. The theoretical study and numerical experiments applied to simulated time series provide comparison of results achieved for different number of decomposition levels and wavelet functions, different threshold values and global or local thresholding for selected thresholding functions. Final results are presented both in numerical and graphical forms.

1 Introduction

Wavelet transform represents a very efficient mathematical tool for one-dimensional or multi-dimensional signal analysis and processing. The paper is devoted to the brief description of wavelet functions used for signal analysis at first. The main part of the paper presents a general algorithm for signal and image decomposition and reconstruction applied for their de-noising at first. The final part of the paper presents the use of signal de-noising for the iterative recovery of signal missing regions.

Mathematical analysis and numerical experiments are devoted to the study of different wavelet functions and to the optimal choice of a thresholding function and its local or global application during the process of signal de-noising and interpolation of its missing values.

2 Principles of Signal Wavelet Analysis

Signal wavelet decomposition using wavelet transform (WT) provides an alternative to the short-time Fourier transform (STFT) for signal analysis [5, 3] resulting in signal decomposition into two-dimensional function of time and scale.

Wavelet functions used for signal analysis are derived from the initial function $W(t)$ forming basis for the set of functions

$$W_{m,k}(t) = \frac{1}{\sqrt{a}}\, W\,(\frac{1}{a}\,(t-b)) = \frac{1}{\sqrt{2^m}}\, W\,(2^{-m}t - k) \tag{1}$$

for discrete parameters of dilation $a = 2^m$ and translation $b = k\ 2^m$. Wavelet dilation closely related to its spectrum compression enables local and global signal analysis. Selected examples of analytically defined wavelet functions are presented in Fig. 1.

3 Wavelet Decomposition and Reconstruction

The principle of signal and image decomposition and reconstruction using wavelet transform is presented in Fig. 2 for an image matrix $[g(n,m)]_{N,M}$. Any one-dimensional signal $\{x(n)\}_{n=1}^{N}$ can be considered as a special case of an image having one column only. The decomposition stage

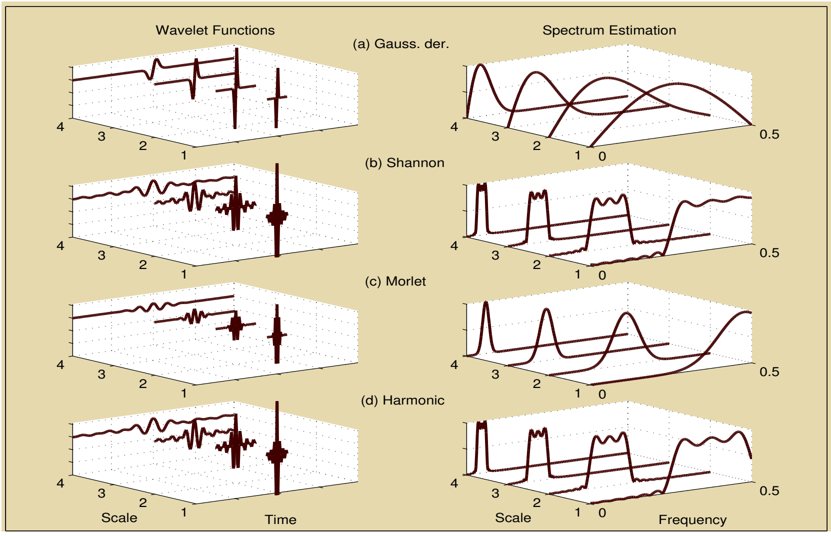


Figure 1: The set of wavelet functions and the effect of their dilation to spectrum compression

includes the processing of matrix $[g(n,m)]_{N,M}$ by columns at first using wavelet (high-pass) and scaling (low-pass) functions in stage $D.1$. Let us denote a selected column of the image matrix $[g(n,m)]_{N,M}$ as signal $\{x(n)\}_{n=0}^{N-1} = [x(0), x(1), ..., x(N-1)]'$. This signal can be analyzed by a half-band low-pass filter with its impulse response

$$\{l(n)\}_{n=0}^{L-1} = [l(0), l(1), \cdots, l(L-1)] \tag{2}$$

and corresponding high-pass filter based upon impulse response

$$\{h(n)\}_{n=0}^{L-1} = [h(0), h(1), \cdots, h(L-1)] \tag{3}$$

The first stage of signal decomposition assumes the convolution of a given signal and the appropriate filter coefficients for decomposition by relations

$$x_l(n) = \sum_{k=0}^{L-1} l(k)\; x(n-k) \qquad\qquad x_h(n) = \sum_{k=0}^{L-1} h(k)\; x(n-k) \tag{4}$$

for all values of n followed by subsampling by factor D.

In the case of signal analysis this procedure is applied to one column of matrix $[g(n,m)]_{N,M}$ only. Studying images this algorithm is then applied to rows of the image matrix followed by column downsampling in stage $D.2$.

The decomposition stage results in this way in two time series (in case of signal processing) or four images representing all combinations of low-pass and high-pass initial image matrix processing. The reconstruction stage in the case of image processing includes row upsampling by factor U at first and row convolution in stage $R.1$ followed by summation of corresponding images. The final step $R.2$ common both for one-dimensional and two-dimensional signals assumes column upsampling and convolution with reconstruction filters followed by summation of the results again.

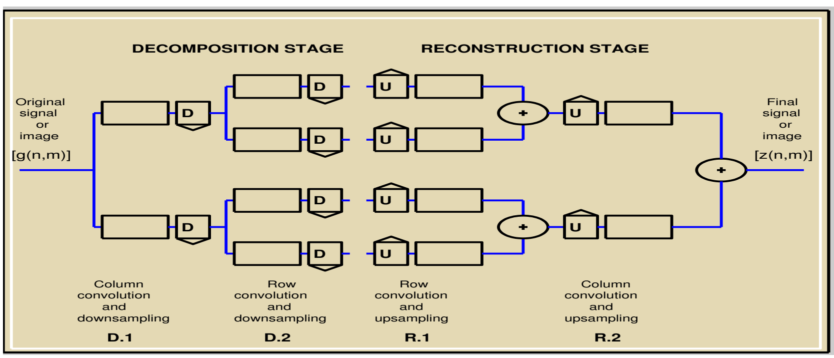


Figure 2: The principle of signal or image decomposition and reconstruction by wavelet transform

4 Signal De-Noising

Both in the case of one-dimensional and two-dimensional signal wavelet decomposition it is possible to modify resulting coefficients $\{c(k)\}_{k=1}^{N}$ before the following signal reconstruction to eliminate undesirable signal components. Methods of such a process assume estimation of appropriate threshold limits [6] and their application to wavelet transform coefficients.

In the case of soft thresholding it is possible to evaluate new coefficients $\{cd(k)\}_{k=1}^{N}$ using original coefficients $\{c(k)\}_{k=1}^{N}$ for a chosen threshold limit δ by the following commands

```
for k=1:N
    if(abs(c(k))<=delta
        cd(k)=0;
    else
        cd(k)=sign(c(k)).*(abs(c(k))-delta);
    end
end
```

Fig. 3 provides an example of the process of signal de-noising by a selected wavelet function applied to a simulated signal presented in Fig. 3(a) defined by a harmonic function with its several

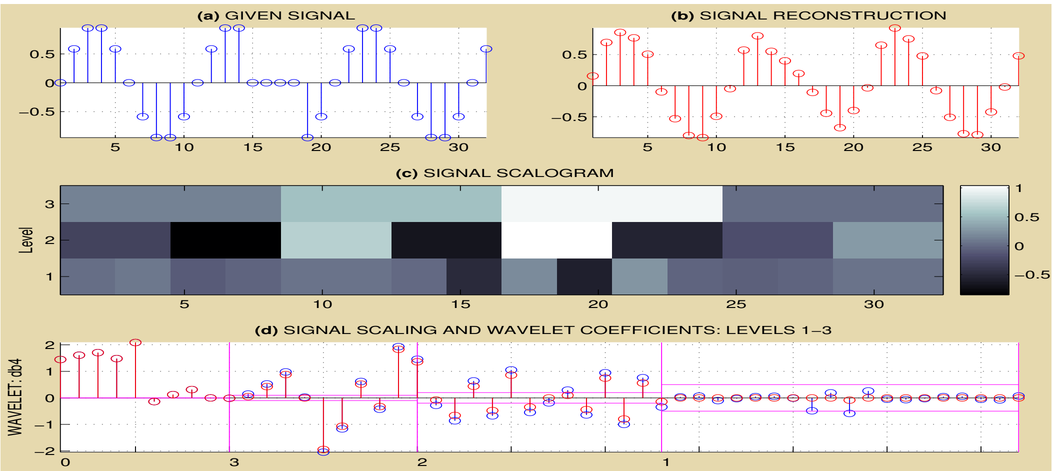
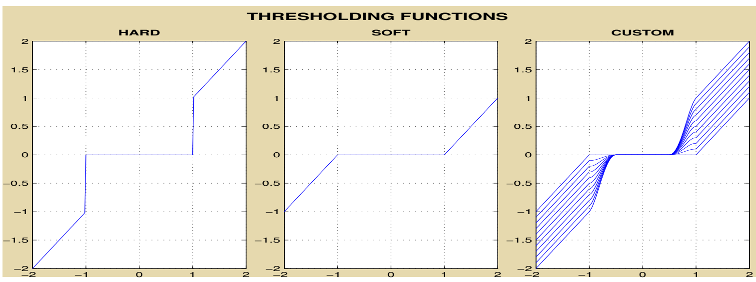


Figure 3: Simulated signal de-noising presenting **(a)** given signal, **(b)** reconstructed signal, **(c)** signal decomposition into two levels, and **(d)** wavelet coefficients thresholding

missing values. This signal is decomposed providing its wavelet transform coefficients $\{c(k)\}_{k=1}^{N}$ visualized by the scalogram in Fig. 3(c) and their values organized in the vector $\mathbf{c}$ presented in Fig. 3(d). Reconstructed signal presented in Fig. 3(b) has been obtained after the application of a selected threshold function (Fig. 4).



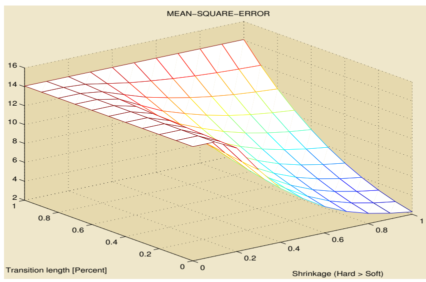


Figure 4: Hard and soft thresholding functions in comparison with custom functions allowing the selection of the transition period and shrinkage level and their optimization to minimize the error between the original and de-noised signal

Figure 5: The study of the effect of different transition period and shrinkage level to the efficiency of signal de-noising

Different kinds of thresholding functions presented in Fig. 4 include hard, soft and custom thresholding functions [8] providing the possibility of the choice of the transition lengths and shrinkage level. Such a possibility can improve the whole process of signal de-noising enabling optimal selection of a method for modification of wavelet coefficients.

Fig. 5 presents an example of the effect of the transition period and shrinkage level to the mean square error between original signal and its de-noised version presented in Fig. 3. Optimal values of these coefficients point to the soft thresholding providing the best results in this case.

Problems closely related to signal de-noising include selection of wavelet decomposition functions, the choice of the thresholding function to modify wavelet coefficients, the decision whether to apply global thresholding using the same thresholing function for all decomposition levels or local thresholding allowing differenciation and mathematical analysis and optimization of thresholding coefficients.

5 Signal Regions Recovery

The recovery of missing or corrupted signal or image regions represents an important problem in many applications and there are many possible approaches how to study this problem.

The wavelet transform represents one possible mathematical tool that can be used to interpolate missing values. The whole algorithm consists of the following iterative steps

- Signal wavelet decomposition into a selected level
- Thresholding of resulting coefficients
- Signal reconstruction
- Replacement of signal values outside the corrupted region by original values
- The following wavelet decomposition and repetition of the whole process

Fig. 3 presents an example of the first step of corrupted signal region recovery assuming its decomposition, thresholding and reconstruction resulting in values presented in Fig. 3(b).

The complete iterative algorithm changes just coefficients of the lost region using thresholding method while all other values are preserved in each step. The algorithm is repeated until the sum of squared errors (SSE) between the recovered and the original signal is acceptably low or required PSNR [dB] is achieved. Fig. 6 presents results of such a process after 20 iterations.

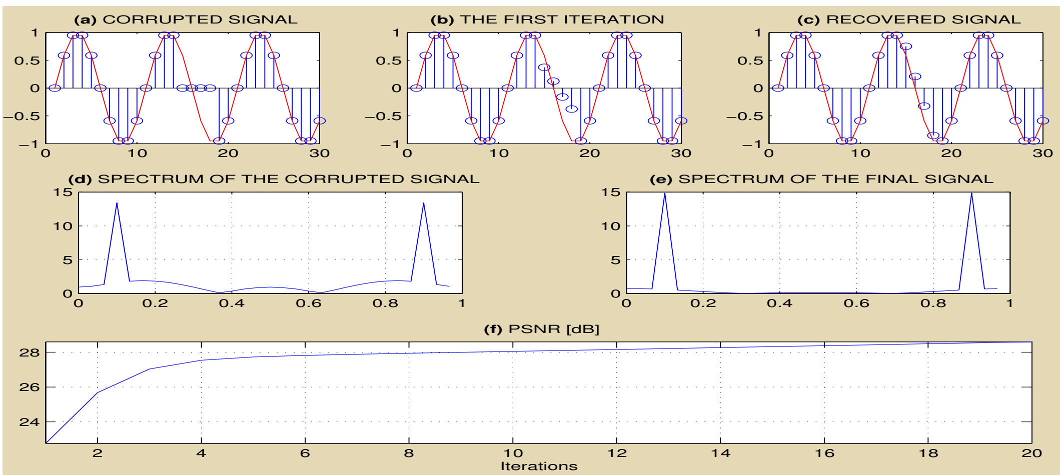


Figure 6: Simulated signal recovery process presenting **(a)** corrupted signal, recovered signal after **(b)** the first and **(c)** the final iteration of the recovery algorithm, spectrum estimation of the **(d)** initial and **(e)** the final signal (after 20 iterations), and **(f)** the peak-signal-to-noise-ratio evolution

Fig. 7 presents the algorithm for definition of thresholding functions selected by the value of parameter $'sorh'$ enabling both hard, soft and custom thresholding for a given limit $'th'$ and selected values of the transition period b and shrinkage a.

```
  function y=LWTthresh(x,sorh,th,b,a)
% Thresholding function
% x - given sequence
% sorh - threshold type
% ('s'-soft,'h'-hard,'v'-vaidyanathan)
% th - threshold limit
% a,b- threshold function constants
% b-relative length of transition (=0-1)
% a-shrinkage from hard (=0) to soft (=1) thresholding
  switch sorh
  case 's'
   tmp=(abs(x)-th); tmp=(tmp+abs(tmp))/2;
   y =sign(x).*tmp;
  case 'h'
   y =x.*(abs(x)>th);
  case 'v'
   th0=(1-b)*th; D=th-th0;
   yc1=(x.*(1-a).*((abs(x)-th0)./D).^2.*...
        (-(a+2)*(abs(x)-th0)./D+3+a)).*(abs(x)>th0 & abs(x)<th);
   yc2=(x-sign(x).*a*th).*(abs(x)>=th);
   y=yc1+yc2;
  otherwise
   error('Invalid argument value')
  end
```

Figure 7: Definition of a function enabling hard, soft and custom thresholding selected by its parameter $'sorh'$ for a given threshold $'th'$ and coefficients a, b used for custom thresholding

Basic commands for wavelet transform de-noising using Matlab notation are summarized in Fig. 8. Fundamental functions used in this program segment include the following:

(i) $[c, l] = wavedec(s, level, wavelet)$ – wavelet decomposition of signal s up to the given $level$ for selected $wavelet$ function

```
% Signal denoising
% Given signal definition
  N=32; n=0:N-1; sref=sin(2*pi*0.1*n);
  s=sref+1*rands(1,N);
% The decomposition to a given level
  level=3; wavelet='db2';
  [c,l]=wavedec(s,level,wavelet);
% Modification of Wavelet coefficients
  delta=thselect(c(l(1):end),'sqtwolog')
  cd=LWTthresh(c,'s',delta);
% Signal reconstruction
  z=waverec(cd,l,wavelet);
```

Figure 8: Basic commands for wavelet transform de-noising after signal decomposition into a selected $level$ $(= 3)$ and $wavelet$ $(=' db2')$ using estimation of a global threshold limit by function $sqrt(2 * log(length(c(l(1) : end)))$ selected by parameter $'sqtwolog'$

(ii) $delta = thselect(x, TPTR)$ – estimation of a threshold limit for a sequence x using selected method defined by parameter $TPTR$

(iii) $z = waverec(cd, l, wavelet)$ – wavelet reconstruction using coefficients cd divided by indices in variable l applying a selected *wavelet* function

These functions enabling signal wavelet decomposition, coefficients thresholding and reconstruction can be replaced by Matlab function *wden* using proper parameters defining the type of thresholding and wavelet decomposition and reconstruction function.

Methods presented above has been verified for simulated signals at first and then applied to real signals and images. An example of this process applied to a selected real signal de-noising is given in Fig. 9 presenting gas consumption in the Czech republic.

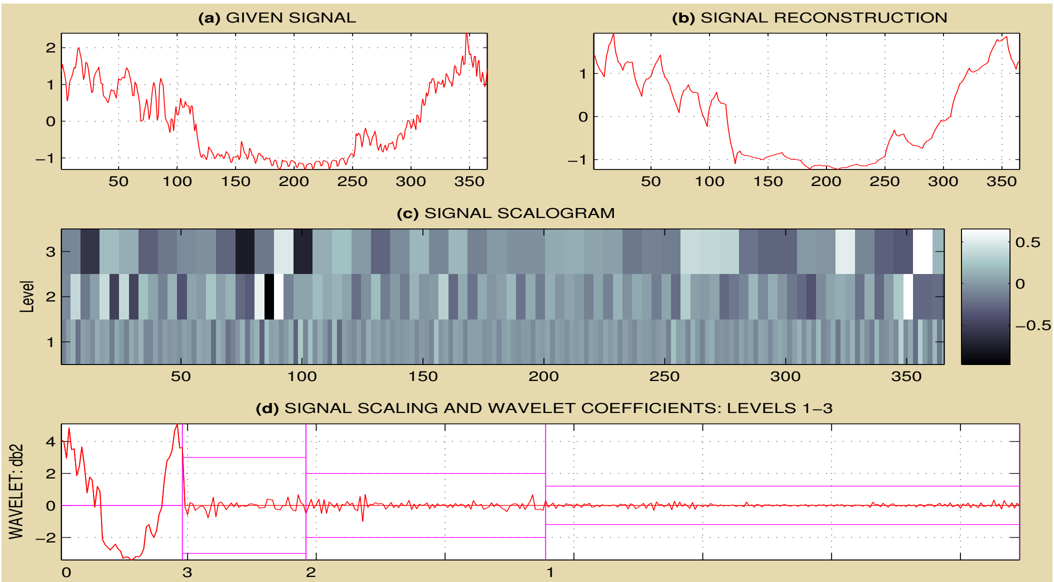


Figure 9: Principle of a gas consumption signal decomposition, thresholding and reconstruction

The iterative method of wavelet regions recovery has can be further modified and used for image processing. Selected result of the recovery of degraded parts of a magnetic resonance image is presented in Fig. 10. Efficiency of the recovery process provides good enough results even though they depend upon the degree and extension of corrupted image region and the choice of wavelet function.

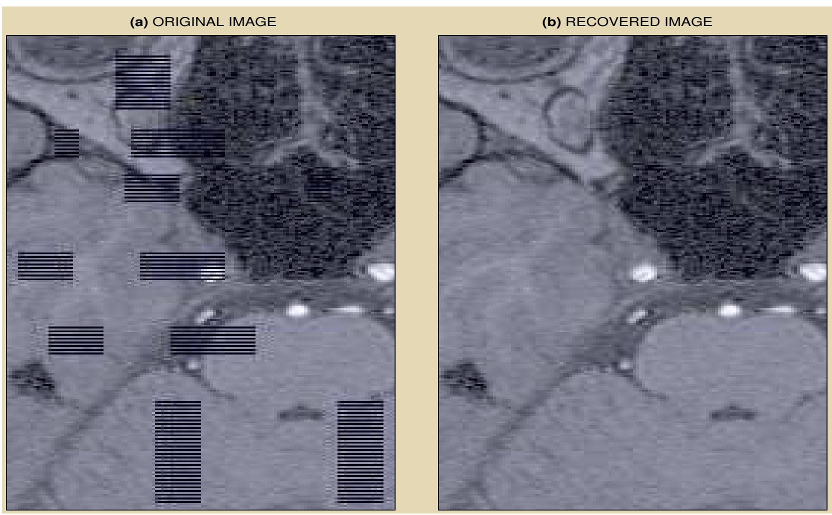


Figure 10: Recovery of a real biomedical image of the brain presenting **(a)** given corrupted image and **(b)** recovered image

6 Conclusion

The paper presents similar approach to signal and image de-noising [7] and recovery of its missing or corrupted values using wavelet transform and providing comparison of different wavelet functions and thresholding methods. Resulting signal can be then used for its further analysis or prediction of its values [2]. In the case of the image this approach can be used for reconstruction of missing parts of images and to more precise classification of their regions [4, 1].

Acknowledgments

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