Application of Multiple Model LQ Control

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Abstract

The optimal control strategy for discrete time multiple model was described. Simulation of control and on-line estimation of model probability is shown in this paper. Comparison of the classical LQ control and LQ control based on multiple model is presented.

Keywords: Control, Estimation, Multiple model, Riccati equation, Simulation.

1 Introduction

In different operating points there are different models of the real plant. The main idea of this paper is optimal control of such set of models.

The methodology based on bayesian update of the probability distribution over the set of possible models [3] enables description of a plant by a mixture distribution [6], [1].

The Bayesian approach to detection of the active model (estimation of optimal model probability) and LQ (optimal control of Linear system with Quadratic criterion) algorithm based on a mixture of a set of parallel models with common state and different structure were described in [2]. Design of such LQ controller and comparison with classical LQ controller is presented in this paper.

LQG (Gaussian noise) algorithm based on a mixture of a set of parallel models with different parameters and different dimension was developed in [5]. Design of such LQ controller with a set of Kalman filters [4] for on-line estimation of model probability is also presented in this paper.

The outline of the paper is as follows: in section 2, the result formulas for LQ controller based on a mixture of a set of parallel models design is noted. In section 3, several LQ controllers based on multiple model is designed for simple examples are presented. Comparison with classical LQ control is shown.

2 Multiple model control

The LQ controller design based on a multiple model is developed [2], [5]. In this section, the main formulas for such LQ controller design are described.

2.1 State feedback controller

Suppose a set of h state development particular models with common state $x(t)$ is given

$$x\,(t+1) = A_i x\,(t) + B_i u\,(t) + v_i\,(t) \tag{1}$$

where A_i, B_i are state matrices of $i-th$ model [4] and $\Gamma_{v_i}(t) = \mathrm{cov}\,\{v_i(t)\}$.

The optimal control law is

$$\begin{aligned} u^*(t) \;=\; & -\left(R(t)+\sum_{i=1}^{h}\alpha_i B_i^T P(t+1) B_i\right)^{-1} \times \\ & \times \sum_{i=1}^{h}\alpha_i B_i^T P(t+1) A_i x(t), \end{aligned} \tag{2}$$

where $Q(t)$, $R(t)$ are criterion matrices [2] and $\alpha_i(t)$ is optimal probability of i–th model. Matrix $P(t)$

is solution of the special Riccati equation

$$\begin{aligned} P(t) &= Q(t)+\sum_{i=1}^{h}\alpha_i A_i^T P(t+1)A_i - \\ &- \left(\sum_{i=1}^{h}\alpha_i A_i^T P(t+1)B_i\right)\times \\ &\times\left(R(t)+\sum_{i=1}^{h}\alpha_i B_i^T P(t+1)B_i\right)^{-1}\times \\ &\times\left(\sum_{i=1}^{h}\alpha_i B_i^T P(t+1)A_i\right) \end{aligned} \tag{3}$$

with final condition $P(N) = Q(N)$.

Note that the optimal feedback gain matrix equals

$$\begin{aligned} K(t) &= \left(R(t)+\sum_{i=1}^{h}\alpha_i B_i^T P(t+1)B_i\right)^{-1}\times \\ &\times\sum_{i=1}^{h}\alpha_i B_i^T P(t+1)A_i \end{aligned} \tag{4}$$

and finally the optimal feedback control law is

$$u^*(t) = -K(t)\,x(t). \tag{5}$$

2.2 Output feedback controller

In this case, the optimal feedback control law is

$$\begin{aligned} u^*(t) &= -\left(R(t)+\sum_{j=1}^{h}\alpha_j B_j^T P_j(t+1)B_j\right)^{-1}\times \\ &\times\sum_{i=1}^{h}\alpha_i B_i^T P_i(t+1)A_i\widehat{x}_i(t|t-1), \end{aligned} \tag{6}$$

where $\widehat{x}_i(t|t-1)$ is the optimal state estimation provide by the $i-th$ Kalman Filter [4] and the special Riccati equation for $P_i(t)$, starting with $P_i(N) = Q_i(N)$ reads

$$\begin{aligned} P_i(t) &= Q_i(t) + A_i^T P_i(t+1)A_i - \\ &- A_i^T P_i(t+1)B_i\times \\ &\times\left(R(t)+\sum_{j=1}^{h}\alpha_j B_j^T P_j(t+1)B_j\right)^{-1}\times \\ &\times B_i^T P_i(t+1)A_i. \end{aligned} \tag{7}$$

Note that the optimal feedback gain matrices equal

$$\begin{aligned} K_i(t) &= \left(R(t)+\sum_{j=1}^{h}\alpha_j B_j^T P_j(t+1)B_j\right)^{-1}\times \\ &\times B_i^T P_i(t+1)A_i \end{aligned} \tag{8}$$

and finally the optimal feedback control law is

$$u^*(t) = -\sum_{i=1}^{h}\alpha_i K_i(t)\,\widehat{x}_i(t|t-1). \tag{9}$$

Note that the Riccati equations (7) and feedback gain matrices (8) cannot be computed separately for each model. More details were described in [5].

3 Simulation results

3.1 Example 1

Consider simple SISO system of second order

$$P(s) = \frac{Y(s)}{U(s)} = \frac{1}{(1+s\tau)^2} \tag{10}$$

with time constant $\tau \in \langle 20; 50\rangle\, s$.

The state space description of system (10) is

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1/\tau & 1/\tau \\ 0 & -1/\tau \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/\tau \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)\,. \end{aligned} \tag{11}$$

Note that it is necessary to use the cascade form for description of the set of models with time constant $\tau \in \langle 20; 50\rangle\, s$.

System (11) is approximated by the set of four models with matrices

$$A_i = \begin{bmatrix} -1/\tau_i & 1/\tau_i \\ 0 & -1/\tau_i \end{bmatrix},\ B_i = \begin{bmatrix} 0 \\ 1/\tau_i \end{bmatrix}, \tag{12}$$

where $\tau_1 = 20s$, $\tau_2 = 25s$, $\tau_3 = 30s$ and $\tau_4 = 50s$.

The LQ control law (2), (3) for reference tracking [4] is designed for the mixture of four models with parameters α_i where $i = 1, 2, 3, 4$. The criterion matrices are $Q = 100$ and $R = 1$.

For the simulation, the time constant of the system (11) is changed from $\tau = 25s$ to $\tau = 50s$ at time $t = 180s$, and next to $\tau = 30s$ at time $t = 340s$, to $\tau = 20s$ at time $t = 540s$, to $\tau = 30s$ at time $t = 680s$ and finally to $\tau = 25s$ at time $t = 840s$. The reference tracking and the optimal estimation of model probability α_i are shown in Figure 1. Note that the change of reference provide sufficient excitation of system for probability distribution tracking.

For robust stability analysis, the nominal model is chosen from the set of models (10). In this case, the nominal time constant is chosen $\tau_n = 35s$. This nominal model is used for classical LQ controller $K_c(\tau_n)$ design. LQ controller based on multiple model $K_m(\alpha^*)$ is designed for optimal model probability which is obtained from Figure 1b. Criterion matrices are

$$Q = \begin{bmatrix} 10^5 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R = 1\,. \tag{13}$$

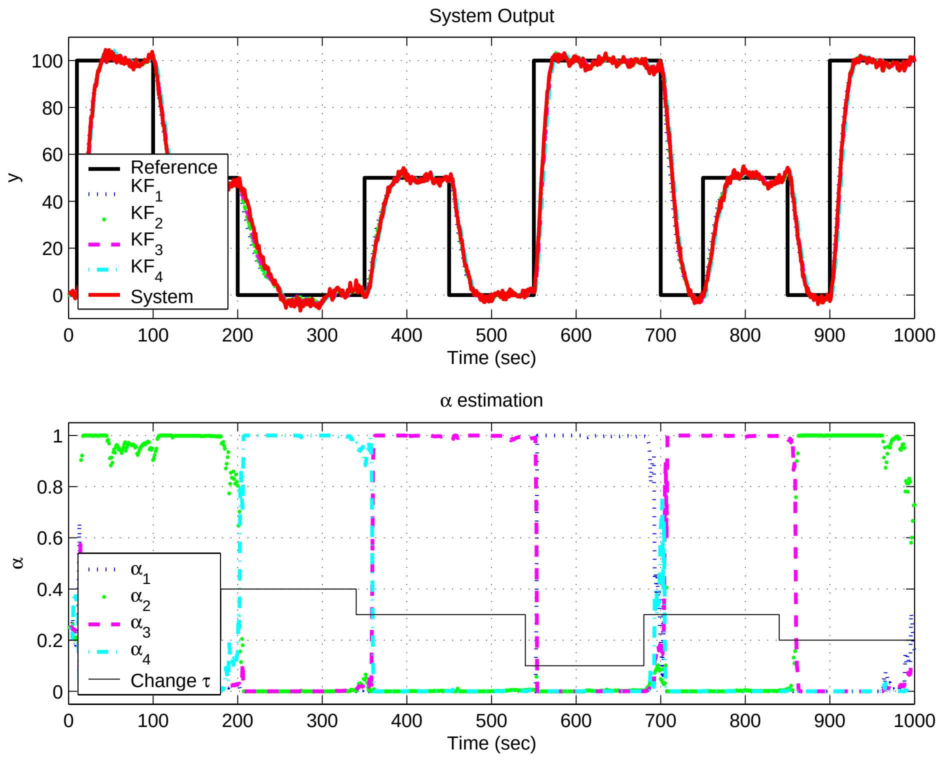

Figure 1: Reference tracking and model probability estimation

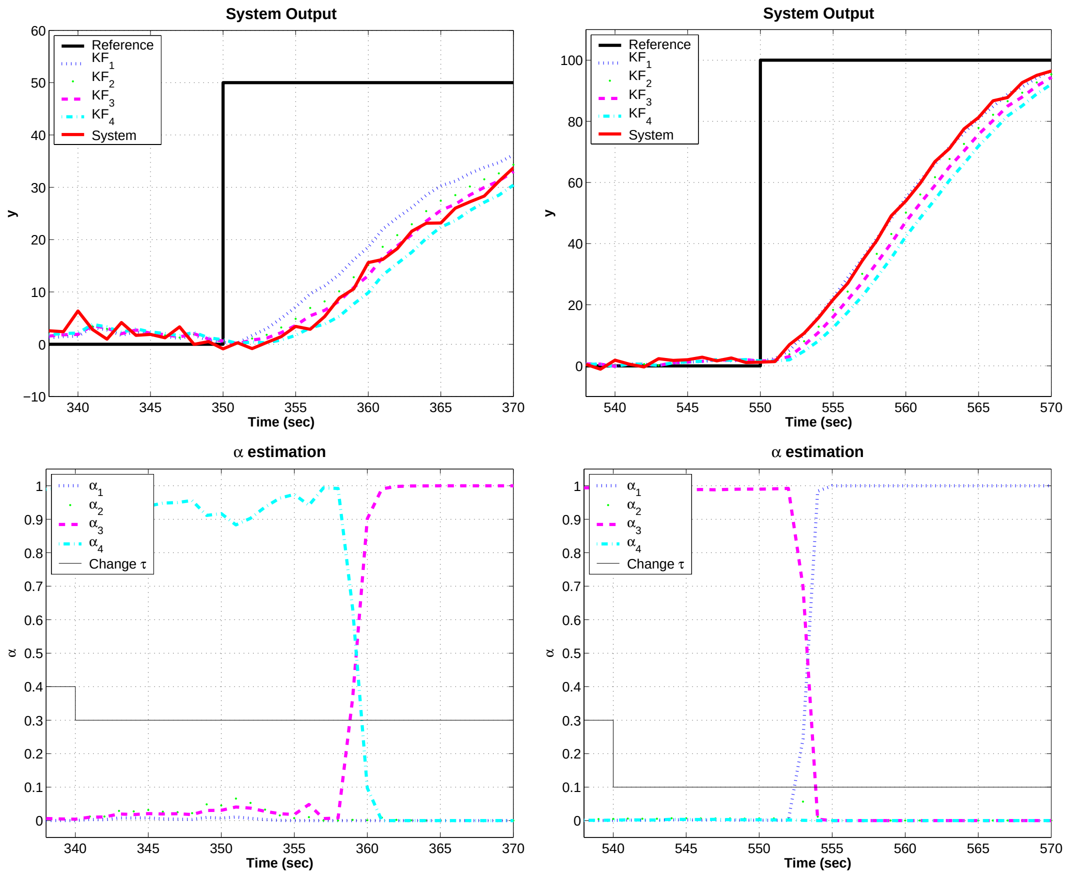

Figure 2: Zoom of Figure 1 around time 350 and 550 seconds

For analysis of the robustness, the time constant of the real system (11) is changed from the set $\tau = \{20s, 25s, 30s, 50s\}$. The eigenvalues of matrices

$$
\begin{aligned}
A(\tau) &- B(\tau)K_c(\tau_n) \quad (14)\\
A(\tau) &- B(\tau)K_m(\alpha^*) \quad (15)
\end{aligned}
$$

are shown in Figure 3.

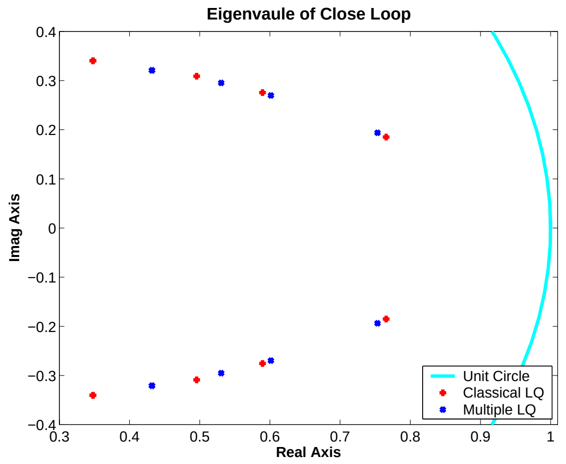

Figure 3: Eigenvalues of close loop

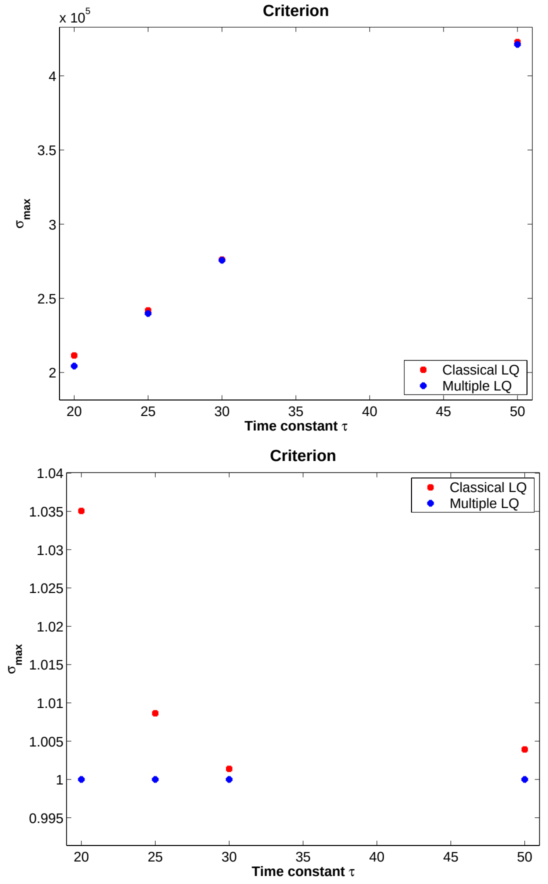

Figure 4: Maximum singular values of close loop

The maximum singular values ($\mathcal{H}_2$ norm) of matrices $P_c(\tau)$ and $P_m(\tau)$ are shown in Figure 4. Matrix $P_c(\tau)$ is the solution of discrete Lyapunov equation

$$
\begin{aligned}
&P_c(\tau) = Q + K_c^T(\tau_n) R\, K_c(\tau_n) + \quad (16)\\
&+ \big[A(\tau) - B(\tau)K_c(\tau_n)\big]^T P_c(\tau) \big[A(\tau) - B(\tau)K_c(\tau_n)\big]
\end{aligned}
$$

and matrix $P_m(\tau)$ is the solution of equation

$$
\begin{aligned}
&P_m(\tau) = Q + K_m^T(\alpha^*) R\, K_m(\alpha^*) + \quad (17)\\
&+ \big[A(\tau) - B(\tau)K_m(\alpha^*)\big]^T P_m(\tau) \big[A(\tau) - B(\tau)K_m(\alpha^*)\big].
\end{aligned}
$$

From Figure 3 follows that both of LQ controllers is stabilized all the set of systems (12). From Figure 4 follows that values of $\mathcal{H}_2$ norm of LQ strategy based on multiple model is less then $\mathcal{H}_2$ norm of LQ strategy based on single model. But the differences are not so significant.

Note that the Figure 4b is just a normalized version Figure 4a.

3.2 Example 2

Consider the same system as in example 3.1. In this case, system (11) is approximated by the set of two models with matrices (12), where $\tau_1 = 20s$ and $\tau_2 = -5s$. The criterion matrices for LQ control law (2), (3) for reference tracking are $Q = 100$ and $R = 1$.

For the simulation, the time constant of the system (11) is changed from $\tau = -5s$ to $\tau = 20s$ at time $t = 180s$, and next to $\tau = -5s$ at time $t = 340s$, to $\tau = 20s$ at time $t = 440s$ and finally to $\tau = -5s$ at time $t = 540s$. The reference tracking and the optimal estimation of model probability α_i are shown in Figure 5.

In this case, the nominal time constant is $\tau_n = 8s$. This nominal model is used for classical LQ controller $K_c(\tau_n)$ design. LQ controller based on multiple model $K_m(\alpha^*)$ is designed for optimal model probability which is obtained from Figure 5b. Criterion matrices are

$$
Q = \begin{bmatrix} 10^2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R = 1\,, \quad (18)
$$

and the eigenvalues of matrices

$$
\begin{aligned}
A(\tau) &- B(\tau)K_c(\tau_n) \quad (19)\\
A(\tau) &- B(\tau)K_m(\alpha^*) \quad (20)
\end{aligned}
$$

for the set $\tau = \{20s, -5s\}$ are shown in Figure 7.

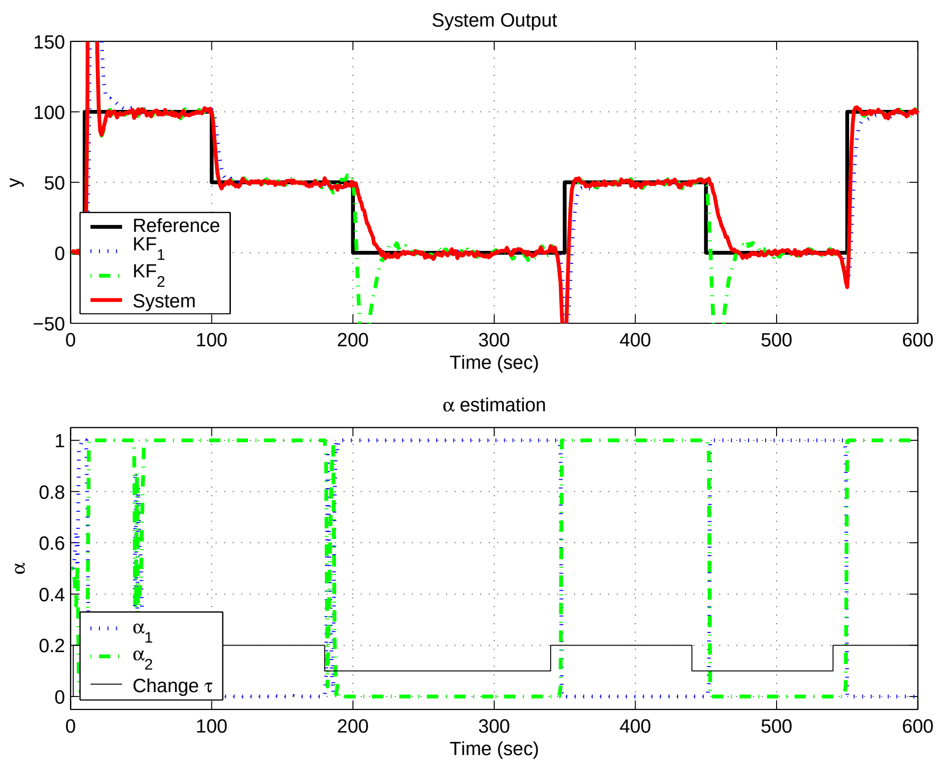

Figure 5: Reference tracking and model probability estimation

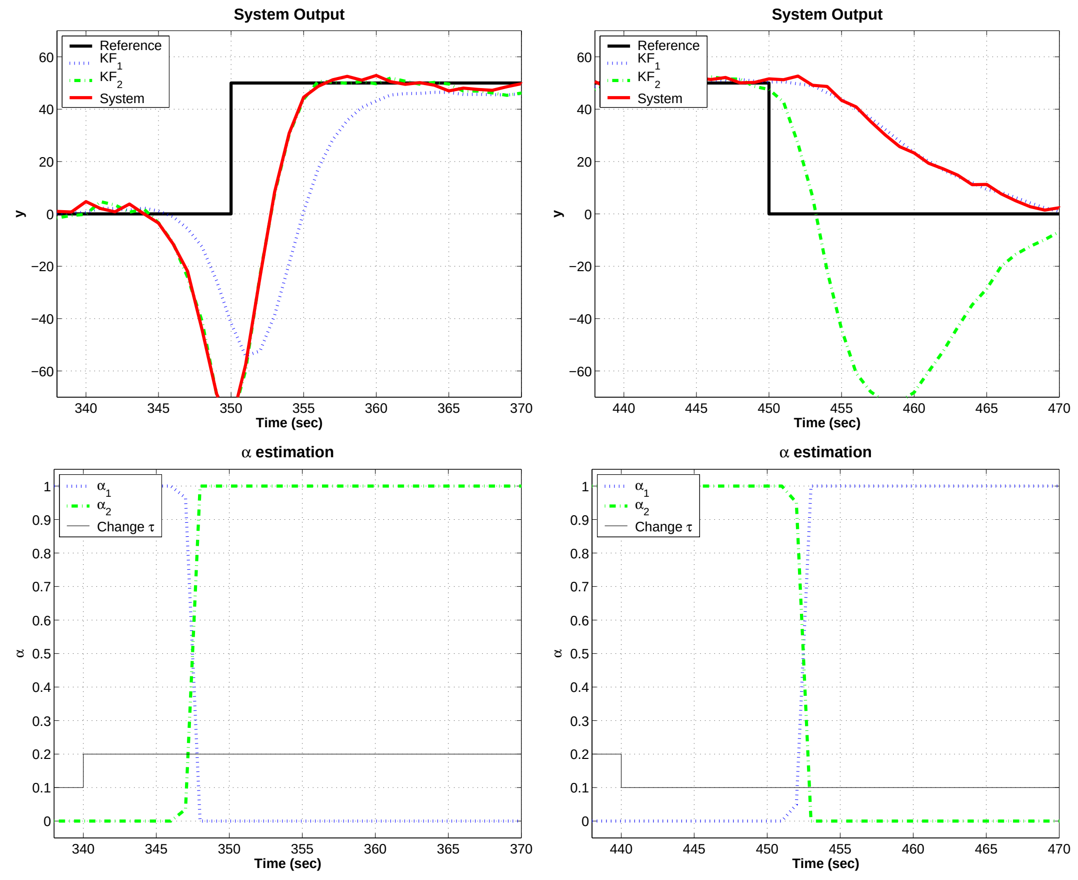

Figure 6: Zoom of Figure 5 around time 350 and 450 seconds

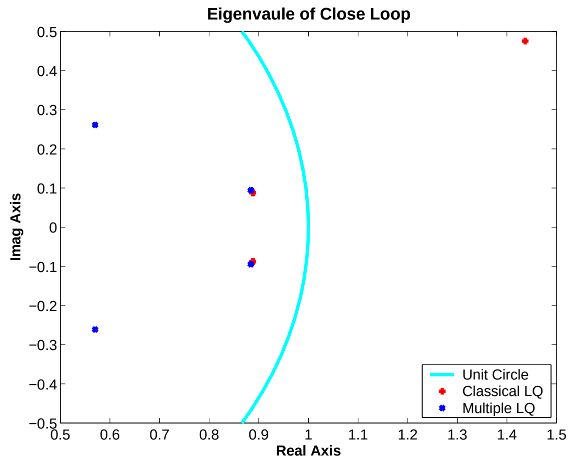

Figure 7: Eigenvalues of close loop

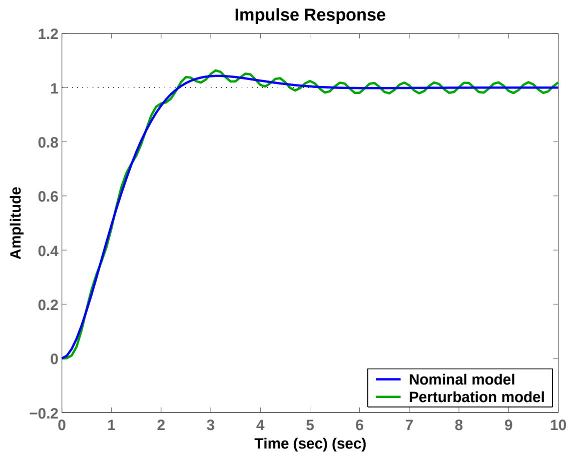

Figure 8: Impulse responses - nominal and perturbation model

Note that unlike classical LQ control, the LQ strategy based on multiple model provide stabilization of system (10) in both case $\tau = \{20s, -5s\}$ (see Figure 7).

3.3 Example 3

SISO system is modelled by mixture of two models with different structure and different dimension.

The nominal model of system is

$$P_0(s) = \frac{1}{0.5s^3 + s^2 + s}\,. \tag{21}$$

It is supposed that system can have poles $-a\,\omega_0 \pm \omega_0\,\sqrt{a^2-1}$ where $a \in (0,1)$ and $\omega_0 = 10$. Then the multiplicative perturbation model is

$$P(s) = P_0(s) \cdot \frac{\omega_0^2}{s^2 + 2\,a\,\omega_0\,s + \omega_0^2}\,. \tag{22}$$

System (22) is approximated by a set of two models with different dimension. The first model $(A_1,\,B_1)$ correspond to the nominal model (21)

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \tag{23}$$

and the second model $(A_2,\,B_2)$ correspond to the worst perturbation model (22) i.e. for $a \to 0$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10 \\ 0 & 0 & 0 & 10 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}. \tag{24}$$

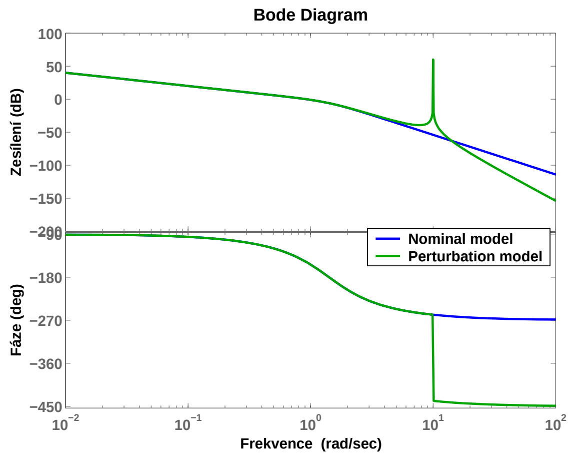

Figure 9: Bode diagrams - nominal and perturbation model

Impulse responses of nominal and perturbation model are in Figure 8. These responses are almost the same and with inexact measurement cannot be distinguished. But the resonance $\omega_0 = 10$ can have essential influence on some controllers. The difference of both models is better seen in Bode diagram (see Figure 9).

For the simulation, the system (24) is changed from $P(s)$ to $P_0(s)$ at time $t = 18s$, and next to $P(s)$ at time $t = 34s$, and finally to $P_0(s)$ at time $t = 54s$.

The probabilities $\alpha_1(t)$ and $\alpha_2(t)$ of models (23) and (24) and the estimation of system states $\widehat{x}_1(t)$ and $\widehat{x}_2(t)$ are provided by two Kalman filters in normalized form [4]. The reference tracking and the optimal estimation of model probability α_i are shown in Figure 11. Criterion matrices for LQ controller based on multiple model design $K_i(\alpha^*)$ (6), (7) are $Q_1 = 500$, $Q_2 = 100$ and $R = 1$.

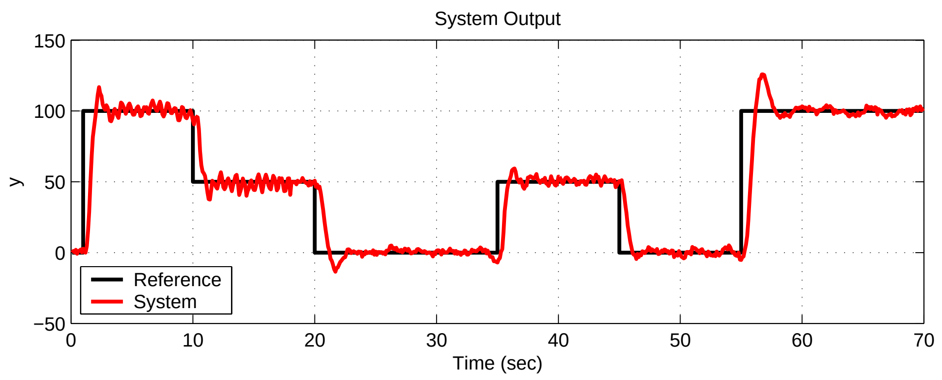

Figure 10: Reference tracking and model probability estimation - Classical LQ

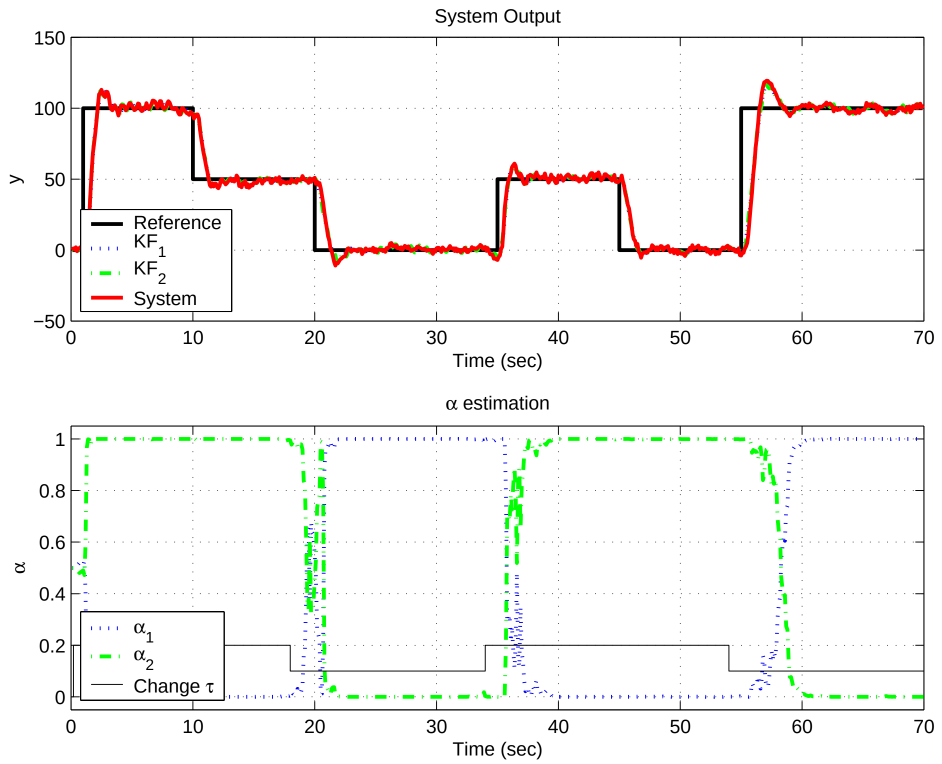

Figure 11: Reference tracking and model probability estimation - Multiple LQ

For nominal model (21), the classical LQ controller is designed for criterion matrices $Q = 500$ and $R = 1$. The reference tracking is shown in Figure 10.

Note that the LQ controller based on multiple model provide better reference tracking mainly when the perturbation system (22) is active (see Figure 11 and Figure 10).

4 Conclusion

The design of LQ controller based on the multiple model and analysis of robustness was presented. The comparison with classical LQ control approach was also presented. Simulation results prove the fact which was expected - multiple model approach is more robust then single model approach. But the differences are not so significant.

Note that the optimal model probability estimation is very sensitive to forgetting coefficient setting and the change of reference provide sufficient excitation of system for probability distribution tracking.

Acknowledgements

This work was partly supported by the Ministry of Education of the Czech Republic under project LN00B096 and by the grant 102/01/0021 of the Grant Agency of the Czech Republic.

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