USING MATLAB FOR SOLVING CONTACT PROBLEM IN ELASTICITY

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Abstract: In geomechanics and biomechanics there are problems whose investigations lead to solving model problems based on variational formulations. Such problems are frequently formulated by variational inequalities as they physically describe the principle of virtual work in its inequality form. This report is devoted to solution variational problems by using MATLAB. In the first part of the contribution we will formulate variational inequality problem, in second part ist finite element approximation and in the third part numerical experiments, e.g. some results on mathematical simulation of total knee joint replacement, will be presented.

1. Formulation of the contact problem

Let the investigated part of the elastic body occupy a union Ω of "s" bounded domains $\Omega^\iota, \iota = 1, ..., s$ in $\mathbb{R}^2$, with Lipschitz boundaries $\partial\Omega^\iota$. Let the boundary $\partial\Omega = \bigcup_{\iota=1}^{s} \partial\Omega^\iota$ consist of four disjoint parts, i.e.

$$\partial\Omega = \overline{\Gamma}_u \cup \overline{\Gamma}_\tau \cup \overline{\Gamma}_c \cup \overline{\Gamma}_o.$$

Let us denote

$$\Gamma_c^{kl} = \partial\Omega^k \cap \partial\Omega^l,\ k, l = 1, \dots, s,\ k \neq l,\ \Gamma_c = \bigcup_{k,l} \Gamma_c^{kl},\ \Gamma_u = \bigcup_{\iota=1}^{s} \Gamma_u^\iota,$$
$$\Gamma_u^\iota = \Gamma_u \cap \partial\Omega^\iota,\ \Gamma_o^\iota = \Gamma_o \cap \partial\Omega^\iota,\ \Gamma_\tau = \bigcup_{\iota=1}^{s} \Gamma_\tau^\iota,\ \Gamma_\tau^\iota = \Gamma_\tau \cap \partial\Omega^\iota.$$

Assume that either

$$\text{meas } \Gamma_c^{kl} > 0 \ \text{ or } \ \Gamma_c^{kl} = \emptyset$$

and either

$$\text{meas } \Gamma_u^\iota > 0 \ \text{ or } \ \Gamma_u^\iota = \emptyset.$$

Let body forces $\mathbf{F}$, surface tractions $\mathbf{P}$ and boundary displacements $\mathbf{u}_0$ are given.

We have the following problem $\mathcal{P}$: find the displacements $\mathbf{u}^\iota$ in all Ω^ι such that

$$\frac{\partial}{\partial x_j}\tau_{ij}(\mathbf{u}^\iota) + F_i^\iota = 0 \ \text{ in } \Omega^\iota,\ \iota = 1, \dots, s,\ i = 1, 2, \tag{1}$$

$$\tau_{ij}(\mathbf{u}^\iota) = c_{ijkm}^\iota e_{km}(\mathbf{u}^\iota) \ \text{ in } \Omega^\iota,\ \mathbf{u}^\iota = \mathbf{u}_0^\iota \ \text{ on } \Gamma_u^\iota, \tag{2}$$

$$u_n^\iota = 0 \ \text{ and } \tau_t^\iota = 0 \ \text{ on } \Gamma_o^\iota,\ \tau_{ij}(\mathbf{u}^\iota)n_j^\iota = P_i^\iota \ \text{ on } \Gamma_\tau^\iota \tag{3}$$

and on every Γ_c^{kl} the following conditions are satisfied:

$$u_n^k - u_n^l \leq 0,\ \tau_n^{kl} \leq 0,\ (u_n^k - u_n^l)\tau_n^k = 0. \tag{4}$$

We denote the stress tensor by τ_{ij}, $e_{ij}(\mathbf{u}^\iota) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$,

$$u_n^k = u_i^k n_i^k,\ u_n^l = u_i^l n_i^k = -u_i^l n_i^l,\ (\text{no sum over } k \text{ or } l),$$
$$u_t^k = u_1^k n_2^k - u_2^k n_1^k,\ u_t^l = u_1^l n_2^k - u_2^l n_1^k,$$
$$\tau_n^k = \tau_{ij}^k n_i^k n_j^k,\ \tau_t^k = (\tau_{ti}^k),\ \tau_{ti}^k = \tau_{ij}^k n_j^k - \tau_n^k n_i^k,\ \tau_t^{kl} \equiv \tau_t^k.$$

In what follows, we introduce

$$W = \prod_{\iota=1}^{s} [H^1(\Omega^\iota)]^2, \quad \|\mathbf{v}\|_W = (\textstyle\sum_{\iota \leq s} \sum_{i \leq 2} \|v_i^\iota\|_{1,\Omega^\iota}^2)^{\frac{1}{2}},$$
$$V_0 = \{\mathbf{v} \in W |\ \mathbf{v} = 0 \text{ on } \Gamma_u \text{ and } v_n = 0 \text{ on } \Gamma_o\},\ V = \mathbf{u}_0 + V_0,$$
$$K = \{\mathbf{v} \in V |\ v_n^k - v_n^l \leq 0 \text{ on } \textstyle\bigcup_{k,l} \Gamma_c^{kl}\}.$$

Assume that $u_{0n}^k - u_{0n}^l = 0$ on $\cup_{k,l}\Gamma_c^{kl}$. Let

$$F_i^\iota \in L^2(\Omega^\iota),\ P_i^\iota \in L^2(\Gamma_\tau^\iota),\ c_{ijkl}^\iota \in L^\infty(\Omega^\iota),\ \mathbf{u}_0 \in W.$$

Definition 1: A function $\mathbf{u}$ is a weak solution of problem $\mathcal{P}$, if $\mathbf{u} \in K$ and

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}) \qquad \forall \mathbf{v} \in K, \tag{5}$$

where

$$a(\mathbf{u}, \mathbf{w}) = \textstyle\sum_{\iota=1}^{s} \int_{\Omega^\iota} c^\iota_{ijkl} e_{ij}(\mathbf{u}^\iota) e_{kl}(\mathbf{w}^\iota)\, dx, \quad L(\mathbf{w}) = \sum_{\iota=1}^{s} (\int_{\Omega^\iota} F^\iota_i w^\iota_i \, d\mathbf{x} + \int_{\Gamma^\iota_\tau} P^\iota_i w^\iota_i \, ds).$$

Remark 2: The problem (5) is equivalent problem to find $\mathbf{u} \in K$ such that

$$\mathcal{L}(\mathbf{u}) = \min_{\mathbf{v} \in K} \mathcal{L}(\mathbf{v}) \tag{6}$$

where $\mathcal{L}(\mathbf{v})$ is quadratic functional defined by

$$\mathcal{L}(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}). \tag{7}$$

2. *Finite element approximation*

Let the domain $\Omega = \bigcup_{\iota=1}^{s} \Omega^\iota$ be approximated by $\Omega_h = \bigcup_{\iota=1}^{s} \Omega^\iota_h$ with polygonal boundary $\partial\Omega_h = \overline{\Gamma}_{uh} \cup \overline{\Gamma}_{\tau h} \cup \overline{\Gamma}_{ch} \cup \overline{\Gamma}_{oh}$, where $\overline{\Gamma}_{uh}, \overline{\Gamma}_{\tau h}, \overline{\Gamma}_{ch}, \overline{\Gamma}_{oh}$ are piecewise linear. Let $\Omega_h = \bigcup_{\iota=1}^{s} \Omega^\iota_h$ be triangulated, let q_i be nodes of used triangulation. Let $\mathcal{T}^\iota_h, \iota = 1, ..., s,$ denote triangulations of polygonal domains $\Omega^\iota_h, \iota = 1, ..., s,$ and $\mathcal{T}_h = \{\mathcal{T}^\iota_h, \iota = 1, ..., s\}$. We assume that $\mathcal{T}^\iota_h, \iota = 1, ..., s,$ are consistent with the respective decompositions of the boundaries $\partial\Omega^\iota_h, \iota = 1, ..., s$ and let the nodes lie on Γ^{kl}_c belonging to the triangulations corresponding to the neighbouring subdomains Ω^k and Ω^l being in a mutual contact. The triangulation $\mathcal{T}_h$ is said to be regular, if all $\mathcal{T}^\iota_h, \iota = 1, ..., s,$ are regular, h is the maximal side of the triangulation. For every node q_i of the triangulation $\mathcal{T}_h$ on Γ^{kl}_c and Γ_o we define the set of indeces $\mathcal{N}^{kl}_i = \{j \in \{1, ..., r\} \mid q_i \in \Gamma^{kl}_{cj}\}$ and $\mathcal{N}_i = \{j \in \{1, ..., r'\} \mid q_i \in \Gamma_{oj}\}$, where $\Gamma^{kl}_c = \bigcup_{j=1}^{r} \Gamma^{kl}_{cj}, \Gamma_o = \bigcup_{j=1}^{r'} \Gamma_{oj}, \Gamma^{kl}_{cj}, \Gamma_{oj}$ denote segments on Γ^{kl}_c, Γ_o and r, r' the number of segments on Γ^{kl}_c and Γ_o, respectively.

Let us define a finite dimensional space V_h by

$$V_h = \{\mathbf{v}_h \mid \mathbf{v}_h \in [C(\Omega^1)]^2 \times \cdots \times [C(\Omega^s)]^2, \mathbf{v}_{h|_{T_{hi}}} \in [P_1(T_{hi})]^2, \forall T_{hi} \in \mathcal{T}_h;$$
$$v_{hn}(q_i) = 0, q_i \in \Gamma_o; \mathbf{v}_h(q_i) = \mathbf{u}_0(q_i), q_i \in \Gamma_u\}$$

and a finite dimensional set of admissible displacements

$$K_h = \{\mathbf{v}_h | \mathbf{v}_h \in V_h,\ (v^k_{hn} - v^l_{hn})(q_i) \leq 0,\ q_i \in \Gamma^{kl}_c\}.$$

Definition 3: Function $\mathbf{u}_h \in K_h$ is a solution of the problem $\mathcal{P}_h$ if

$$a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \geq L(\mathbf{v}_h - \mathbf{u}_h) \qquad \forall \mathbf{v}_h \in K_h. \tag{8}$$

Note that in a general case $K_h \not\subset K$.

The next theorem gives the connection between the problem $\mathcal{P}$ and the problem $\mathcal{P}_h$ if $h \to 0_+$ under the assumption that the solution of the problem is sufficiently smooth.

Theorem 4: Let $\partial\Omega$ and its parts $\Gamma_u, \Gamma_\tau, \Gamma_o, \Gamma_c$ be piecewise polygonal, $\Gamma^{kl}_c = \bigcup_{j=1}^{r} \Gamma^{kl}_{cj}$. Let the solution of problem $\mathcal{P}$ $\mathbf{u} \in K \cap [H^2(\Omega)]^2$, $\tau_{ij}(\mathbf{u}^\iota) \in H^1(\Omega^\iota)$, $i, j = 1, 2$ and $\iota = 1, \ldots, s, \tau^{kl}_n(\mathbf{u}) \in L^\infty(\Gamma^{kl}_c)$, $u^k_n, u^l_n \in H^2(\Gamma^{kl}_c), k, l = 1, \ldots, s$ and $j = 1, \ldots, r$. Let $K_h \subset K$. Let changes $u^k_n - u^l_n < 0 \to u^k_n - u^l_n = 0$ and $u^k_t - u^l_t = 0 \to u^k_t - u^l_t \neq 0$ occur at only finitely many points of $\bigcup_{k,l} \Gamma^{kl}_c$. Then for the semi-coercive case

$$|\mathbf{u} - \mathbf{u}_h| = O(h), \text{ where } |\mathbf{w}| = (\sum_{\iota=1}^{s} \int_{\Omega^\iota_h} e_{ij}(\mathbf{w}) e_{ij}(\mathbf{w})\, d\mathbf{x})^{\frac{1}{2}} \tag{9}$$

and for the coercive case

$$\|\mathbf{u} - \mathbf{u}_h\|_W = O(h). \tag{10}$$

For the proof see [1].

Now we solve the problem $\mathcal{P}_h$. If we do not consider the constraints on Γ_o and Γ_u, we may write for $\mathbf{v}_h \in V_h$,

$$\mathbf{v}_h = (\mathbf{v}_h^1, \mathbf{v}_h^2, \ldots, \mathbf{v}_h^s),\ \mathbf{v}_h^l = (v_{h1}^l, v_{h2}^l),\ 1 \le l \le s,$$

$$v_{hi}^l(\mathbf{x}) = \sum_{j=1}^{M(l)} v_i^l(q_j^l)\varphi_j^l(\mathbf{x}) = \sum_{j=1}^{M(l)} x_{ij}^l \varphi_j^l(\mathbf{x})\ ,\ i = 1,2;\ l = 1, \ldots, s, \tag{11}$$

where q_j^l are the nodes of the triangulation, x_{ij}^l the degrees of freedom, $\varphi_j^l(\mathbf{x})$ the basis functions on V_h such, that

$$\varphi_i^l(q_j^l) = \delta_{ij} \quad i,j = 1, \ldots, M(l),\ l = 1, \ldots, s, \tag{12}$$

and $M(l)$ is the number of nodes in the l-th body.

In regard to (11), (12), the constraints on Γ_0 and Γ_u always bind degrees of freedom x_{ij}^l which belong to one node of the triangulation. The constraints on $\Gamma_c = \cup\Gamma_c^{kl}$ express the relation between the displacements $\mathbf{u}_h^k$ and $\mathbf{u}_h^l$ of the two nodes, which form the contact pair, and each of them belongs to different body ($1 \le k < l \le s$) of the model. Therefore, one constraint binds two pairs of degrees of freedom. For simplicity's sake we denote the nodes in a contact pair by the same symbol.

All constraints can be written as

$$\begin{array}{lcll} x_{i1} & = & \mathbf{u}_{01}(q_i) & q_i \in \Gamma_u, \\ \qquad\qquad x_{i2} & = & \mathbf{u}_{02}(q_i) & q_i \in \Gamma_u, \\ x_{i1}n_1(q_i) + x_{i2}n_2(q_i) & = & 0 & q_i \in \Gamma_0, \\ x_{i1}^k n_1(q_i) + x_{i2}^k n_2(q_i) - x_{i1}^l n_1(q_i) - x_{i2}^l n_2(q_i) & \le & 0 & q_i \in \Gamma_c, \end{array} \tag{13}$$

where $\mathbf{n}(q_i) = (n_1(q_i), n_2(q_i))$ denote the outward unit normal to the boundary in node q_i.

The conditions on Γ_u will be satisfied during the assembling of the stiffness matrix and the right hand side vector, i.e. during the assembling of the functional $\mathcal{L}$. The corresponding degrees of freedom are constant, i.e. they are not dependent. In the conditions on Γ_o one parameter of x_{i1}, x_{i2} can be also expressed by the second one.

For these reasons we may consider only the conditions on Γ_c in what follows. These can be written in a matrix form as

$$\begin{array}{ll} Ax \le 0, & A \text{ is of the type } M \times N, \\ & M \text{ is the number of constraints,} \\ & N \text{ is the number of degrees of freedom in the whole model.} \end{array}$$

We will form $\mathcal{L}$ on particular triangles and edges of the triangulation. Let us introduce the vector 3×1, $\overline{e}_{ij}$, $1 \le i \le j \le 2$, by the relations

$$\begin{aligned} \overline{e}_{ii} &= e_{ii} \\ \overline{e}_{12} &= 2e_{12}, \end{aligned} \tag{14}$$

and $f(x) = \mathcal{L}(x_l\varphi_l) = \mathcal{L}(\mathbf{v}_h),\ x \in \mathbb{R}^N$.

It holds that

$$\sum_{i,j,k,l=1}^{2} c_{ijkl}e_{kl}e_{ij} = \sum_{i\le j\ k\le l\ i,j,k,l=1}^{2} c_{ijkl}\overline{e}_{kl}\overline{e}_{ij},$$

which can be written in the matrix form as $\overline{\mathbf{e}}^T D\overline{\mathbf{e}}$, where the matrix D is 3×3, symmetric.

In regard to the choice of V_h, we seek the vector $\mathbf{u}_h = (u_{h1}, u_{h2})$ in the form of linear polynomial on every triangle T_k and edge B_l of the triangulation. Similarly we will obtain $f_k(x_k)$ on a given element in the form $f_k(x) = \frac{1}{2}x_k^T C_k x_k - x_k^T d_k$, C_k is 6×6, $x_k = (6 \times 1)$, $d_k = (6\times1)$. We will also obtain the contributions from the edges on Γ_τ, $x_l^T h_l$, $x_l = (4\times1)$, $h_l = (4 \times 1)$ which will be added to the linear term of $\mathcal{L}$.

Then, we eliminate the contingent degrees of freedom on Γ_u or Γ_0. During the assembling of $\mathcal{L}$ in the whole model, we follow the global numbering of nodes and the numbering of degrees of freedom (i.e. the numbering of the variables in the functional).

The problem $\mathcal{P}_h$ then leads to the quadratic programming problem $\mathcal{P}_d$:

$$f(x) = \tfrac{1}{2}x^T Cx - x^T d \to \min$$

$$\text{with constraints} \quad Ax \leq 0.$$

Remark 5: The global stiffness matrix C is of the type $N \times N$, block diagonal, every block is sparse, symmetric, positive semidefinite matrix and corresponds to just one body in the model. In the coercive case C is positive definite. The constraint matrix A is of the type $M \times N$, $M \ll N$; we assume its rows to be linearly independent.

3. Numerical experiments

The paper presents two models. The first one represents the loaded total endoprothesis of the knee joint in the sagital cross-section and the second model in frontal cross-section. Both models are presented in Fig. 1.

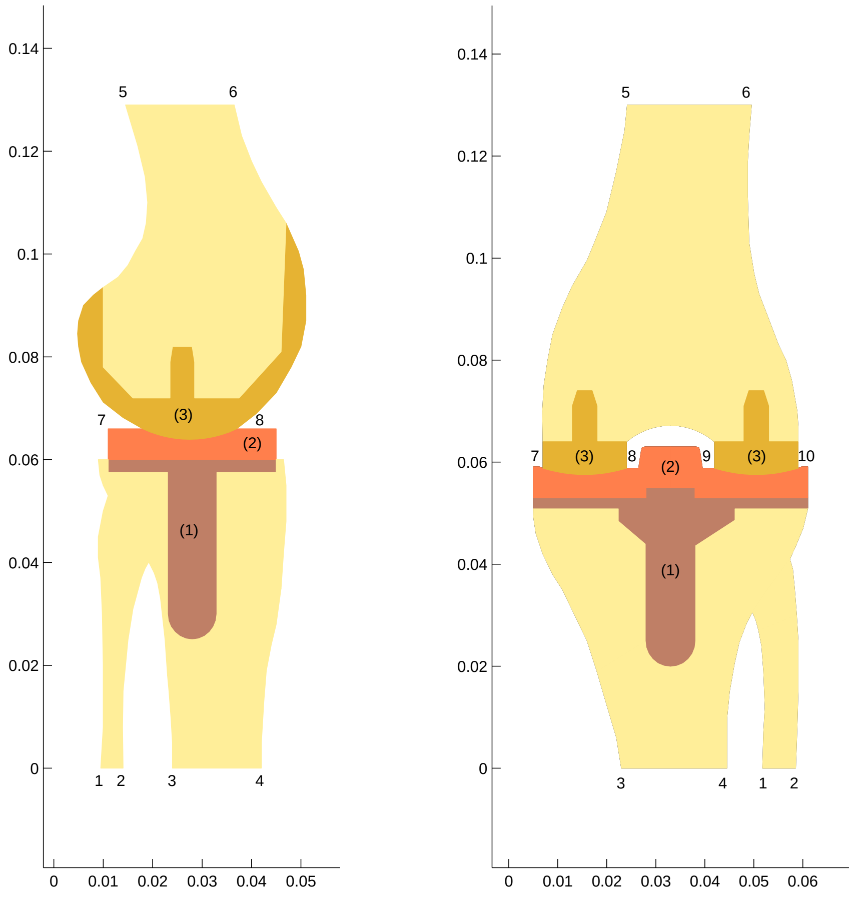
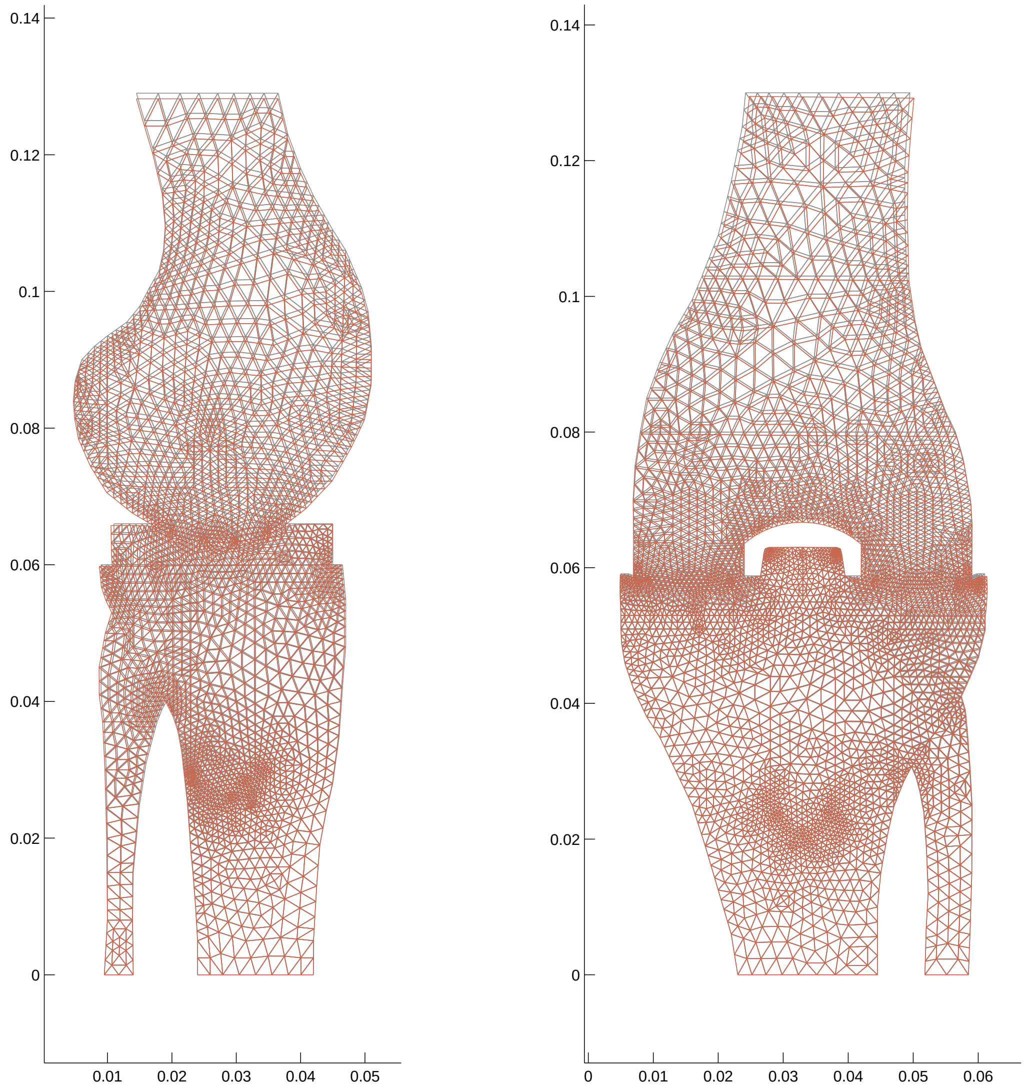


Fig. 1. The model of the artificial knee replacement, (a) the sagital cross-section, (b) the frontal cross-section

The physical parameters are as follows: bone: Young's modulus $E = 1.71 \times 10^{10}$ $[Pa]$, Poisson's ratio $\nu = 0.25$, (1) Ti6Al4V: $E = 1.15 \times 10^{11}$ $[Pa]$, $\nu = 0.3$, (2) chirulen: $E = 3.4 \times 10^8$ $[Pa]$, $\nu = 0.4$, (3) the zircon ceramics $ZrO_2 : E = 4.0 \times 10^{11}$ $[Pa]$, $\nu = 0.22$. The femur is loaded between points 5 and 6 by a loading $0.215 \times 10^7 [Pa]$, the tibia and the fibula are fixed between points 1 and 2 (the tibia) and between 3 and 4 (the fibula) and the unilateral contact boundary is between points 7 and 8 as well as between 9 and 10. The loadings evoked by muscular forces were neglected. In Fig. 2 the deformations and in Fig. 3 the vertical stress tensor components are presented.



(a) (b)

Fig. 2. The deformations (enlarging factor is 10), (a) the sagital cross-section, (b) the frontal cross-section

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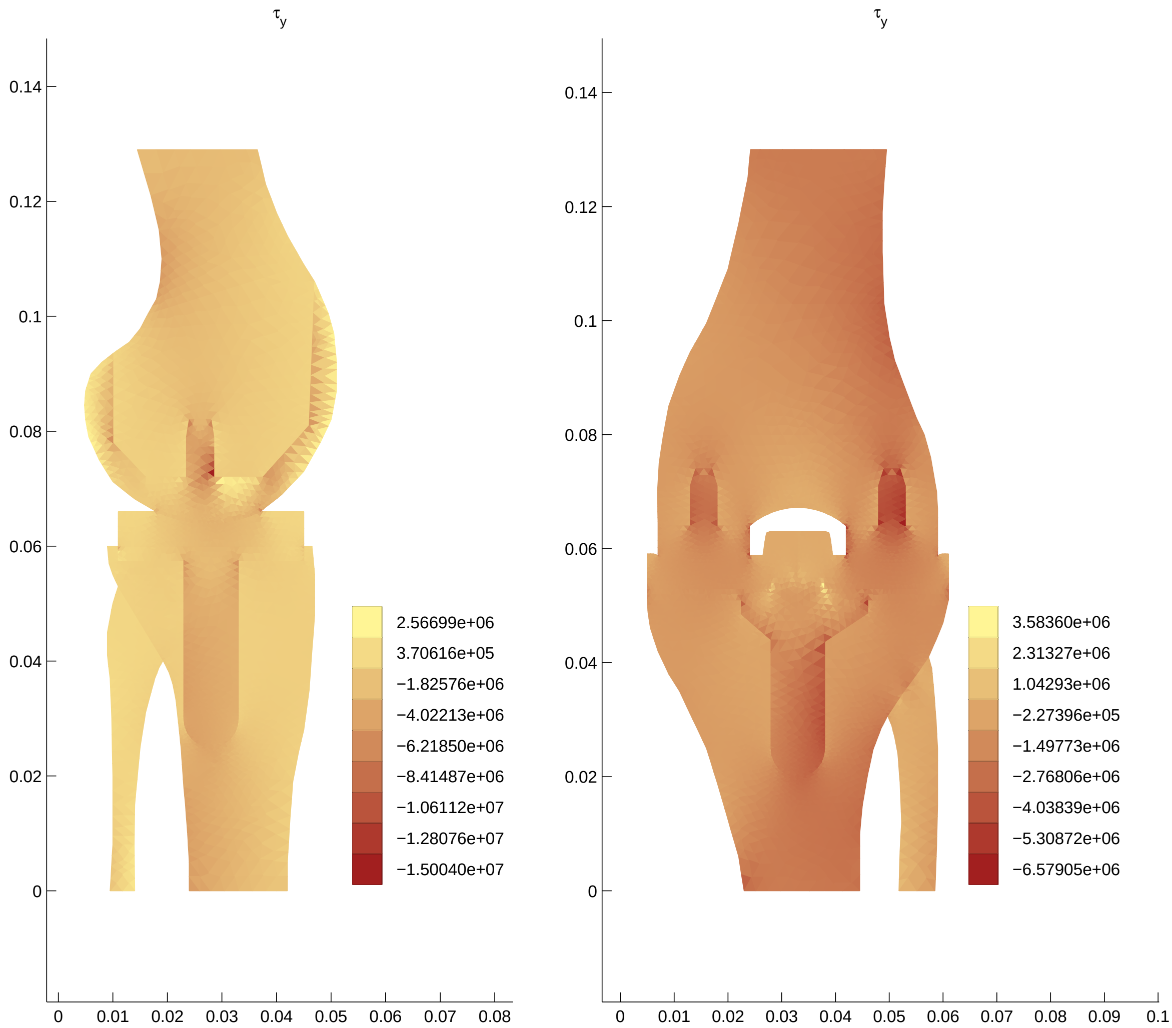

(a) (b)

Fig. 3. The vertical stress tensor components, (a) the sagital cross-section, (b) the frontal cross-section

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