Compromise Discrete Non-linear Control and Its Stability

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Abstract:
The stability of discrete control system can be studied using Lyapunov technique. There is possible to develop a set of the stable control systems in case of given convex Lyapunov function. Let $\vec{x} \in \mathbf{R}^n$ be an initial state, $\vec{x}_1 = \vec{f}_1(\vec{x}) \in \mathbf{R}^n, \ldots, \vec{x}_H = \vec{f}_H(\vec{x}) \in \mathbf{R}^n$ be a *H*-tuple of new states. It is easy to prove that any vector $\vec{x}_{new} \in \mathbf{R}^n$ which is an element of the convex hull C_H of $\{\vec{x}_1, \ldots, \vec{x}_H\}$ will also produce the stable control with the same Lyapunov function. When a new heuristic controller will produce $\vec{x}_h = \vec{f}_h(\vec{x})$, then the projection of $\vec{x}_h$ into given convex hull C_H is a way how to construct a stabile heuristic control system. The function library was realized in the Matlab environment to demonstrate the discrete dynamic.

Discrete Dynamic System

Let $n \in \mathbf{N}$ be the dimension of the *state space*. Let $\vec{x}, \vec{x}^* \in \mathbf{R}^n$ be current and future **state**. Let $\vec{f} : \mathbf{R}^n \to \mathbf{R}^n$ be the system *dynamic* satisfying $\vec{f}(\vec{0}) = \vec{0}$ where $\vec{0} \in \mathbf{R}^n$. Then the *discrete dynamic system* is defined by the rule $\vec{x}^* = \vec{f}(\vec{x})$ or $\vec{x}_{k+1} = \vec{f}(\vec{x}_k)$ respectively, where $k \in \mathbf{N}_0$ is the index of discrete time. The zero vector $\vec{x} = \vec{0}$ is called a *stationary point*.

The aim of this paper is to build up a set of global asymptotically stable systems with the unique stationary point $\vec{x} = \vec{0}$.

Lyapunov Theorem for Discrete Systems

Let $L : \mathbf{R}^n \to \mathbf{R}$ be a continuous function satisfying:

$L(\vec{0}) = 0$

$L(\vec{x}) > 0$ for $\vec{x} \neq \vec{0}$

$L(\vec{x}^*) = L\left(\vec{f}(\vec{x})\right) < L(\vec{x})$ for $\vec{x} \neq \vec{0}$

Then $L(\vec{x})$ is called *Lyapunov function* of the discrete dynamic system $\vec{x}^* = \vec{f}(\vec{x})$ and the stationary point $\vec{x} = \vec{0}$ is globally asymptotically stable. It is not necessary to use smooth Lyapunov function $L(\vec{x})$ for the discrete system investigation. But the convexity of Lyapunov function $L(\vec{x})$ is an advantage in some applications.

Compromise Discrete Systems

Let $H \in \mathbf{N}$ be the number of discrete dynamic systems. Let $\vec{x}_1 = \vec{f}_1(\vec{x}), \ldots, \vec{x}_H = \vec{f}_H(\vec{x})$ are their dynamic descriptions. Let $L(\vec{x})$ be the common convex Lyapunov function for the systems $\vec{f}_1, \ldots, \vec{f}_H$. Let

$$O_H = \left\{ \vec{w} \in \left(\mathbf{R}_0^+\right)^H \middle| \sum_{k=1}^{H} w_k = 1 \right\}$$

be a set of acceptable weights. Let

$$C_H = \left\{ \vec{x} = \sum_{k=1}^{H} w_k \vec{x}_k \mid \vec{w} \in O_H \right\}$$

be a convex hull of $\{\vec{x}_1, \ldots, \vec{x}_H\}$. Then any discrete dynamic system described by the rule $\vec{f} : \mathbf{R}^n \to \mathrm{C}_H$ is also global asymptotically stable.

It means that any vector $\vec{x}^* \in C_H$ belonging to the convex hull of $\{\vec{x}_1, \ldots, \vec{x}_H\}$ can be a new state of global asymptotically stable discrete dynamic system. It is rather typical to use a non-linear heuristics feedback in the real control system. The neural, fuzzy and constrained controllers are good examples of this habit.

Let $\vec{x}_h^* = \vec{f}_h(\vec{x})$ be the dynamic equation of non-linear discrete dynamic system with any heuristics, which is not necessary stable. When $\vec{x}_h^* \in C_H$, there are no problems with the stability with respect to $L(\vec{x})$. In the opposite case when $\vec{x}_h^* \notin C_H$, the stability is not guaranteed.

The potentially unstable control system can be stabilized by any projection of $\vec{x}_h \in \mathbf{R}^n$ into $\mathrm{C}_H \subset \mathbf{R}^n$. Let $\vec{x}^* \in C_H$ be as similar as possible to the original value $\vec{x}_h \in \mathbf{R}^n$. In the mathematical terms it is necessary to find $\vec{x}^* \in C_H$ for which the Euclidean norm $\left\|\vec{x}_h - \vec{x}^*\right\|$ is the minimum one. The value of $\vec{x}^*$ can be found by using the optimum vector $\vec{w}^* \in O_H$. It is equivalent to the quadratic programming task

$$\Phi(\vec{w}) = \left\| \vec{x} - \sum_{k=1}^{H} w_k \vec{x}_k \right\|^2 = \min$$

$w_k \geq 0$ for k=1,…,H

$$\sum_{k=1}^{H} w_k = 1.$$

Let $\vec{w}^* \in O_H$ be optimum solution satisfying $\Phi(\vec{w}^*) \leq \Phi(\vec{w})$ for all $\vec{w} \in O_H$. Now $\vec{x}^* = \sum_{k=1}^{H} w_k^* \vec{x}_k \in C_H$ is the optimum projection to the convex hull C_H.

Matlab Realization

The Matlab environment was used for the realization of the discrete non-linear heuristic but stable system based on the function library consisting of

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L=Lyapunov(x)
X=Dynamics(x)
xnew=Heuristics(x)
xnew=Projection(x)
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Here x is a row vector $(1 \times n)$ of the current state, *xnew* is a row vector $(1 \times n)$ of the new state and X is a matrix $(H \times n)$ composed of H-tuple of new states. The set of four functions enables to design the common convex Lyapunov function, H-tuple of stable discrete dynamic systems, any dynamic system with heuristic controller and the final stable discrete system with projection of $\vec{x}_h$ into C_H.

Numeric Example

Let $n=2$, $H=4$, $\vec{x} = (x_1, x_2)$, $\vec{x}^* = (x_1^*, x_2^*)$, $\vec{x}_h = (h_1, h_2)$, $\vec{x}_k = (x_{k1}, x_{k2})$ for k=1,…,H.

There are four global asymptotically stable systems

$$x_{11} = x_2/2,\ x_{12} = x_1/2$$

$$x_{21} = \sqrt{1 + x_2^2} - 1,\ x_{22} = x_1$$

$$x_{31} = x_2|x_2|/(1 + |x_2|),\ x_{32} = x_1|x_1|/(2 + |x_1|)$$

$$x_{41} = x_1 e^{-|x_2|},\ x_{42} = x_2/(1 + x_1^2)$$

with common Lyapunov function $L = x_1^2 + x_2^2$. Let the heuristic unstable system is described as $h_1 = -x_2,\ h_2 = x_1^2 - |x_2|$. In the case when $\vec{x} = (-1, -1)$ we have

$\vec{x}_1 = (-1/2, -1/2),$
$\vec{x}_2 = (\sqrt{2} - 1, -1),$
$\vec{x}_3 = (-1/2, -1/3),$
$\vec{x}_4 = (-e^{-1}, -1/2),$
$\vec{x}_h = (1,0)$

The point which is an element of C_H and the nearest to $\vec{x}_h$ is $\vec{x}^* = (\sqrt{2} - 1, -1) = \vec{x}_2$ and the projection improves the original heuristic $\vec{x}_h = \vec{f}_h(\vec{x})$. The projection is guaranteed to be global asymptotic stable with $L(\vec{x})$ which is not the property of the heuristic $\vec{x}_h = \vec{f}_h(\vec{x})$. The original value of Lyapunov function $L(\vec{x})$=2 is decreased to the $L(\vec{x}_h)$=1 from the pragmatic point of view by the heuristics but the value of $L(\vec{x}^*) = 4 - 2\sqrt{2} > 1$. The stabilized heuristic is the slower then the original one in the point $\vec{x} = (-1, -1)$.

Conclusions

There is a way how to convert any unstable discrete system to the stable one. Lyapunov function, the set of stable systems and projection into the convex hull are main parts of the solution. The testing example was built to demonstrate the implementation details. The library of Matlab functions was realized as a tool for the computer simulation.

Literature

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