A BACKWARD INTEGRATION METHOD FOR SOLVING DYNAMIC MODELS OF ENVIRONMENTAL POLICY USING MATLAB

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Abstract: this paper uses a backward integration method for solving dynamic environmental models. The formulation of these models typically leads to the integration of a dynamic system, which exhibits saddle-path stability. Therefore standard initial-value integration methods for dynamical systems must not be used, since even a small deviation form the equilibrium manifold leads to the exponential divergence. The backward integration method is based on the change of the sign of the system and the dynamic path is integrated backward from the system steady state. The method is illustrated on a simple model of environmental taxation.

I. Introduction

Dynamic perfect-foresight economic models can often be expressed in the following form:

$$\max \int_0^\infty U(dX/dt, X)e^{-\beta t}dt \tag{1}$$

$$\text{subject to } (X, dX/dt) \in \Gamma(t),\ X(0) \text{ given} \tag{2}$$

Given β>0, and under standard concavity and regularity assumptions, it is known that the unique optimal solution satisfies Euler equations $e^{-\beta t}\left[dU_1/dt - \beta U_1\right] = U_2$ (subject to (2)), along with transversality conditions: $\lim_{t\to\infty} e^{-\beta t}U_2{}^T X(t) = 0$. (Subscripts denote partial derivatives). See Beneviste and Scheinkman (1982) or Kamihigashi (2001) for a rigorous discussion.

However, in a typical case, it is possible to factorize the vector $X(t)$ in so-called state variables $K(t)$ and in control variables $C(t)$ and to write the Euler equations as a following system:

$$\begin{bmatrix} dK/dt \\ dC/dt \end{bmatrix} = \begin{bmatrix} G(K(t), C(t)) \\ H(K(t), C(t)) \end{bmatrix} = F(K(t), C(t)) \tag{3}$$

$K(0)$ given and the transversality condition: $\lim_{t\to\infty} e^{-\beta t}\left(\partial U(C,K)/\partial C\right)^T K(t) = 0$

Note that only for the state variables, initial conditions are given. There are no such conditions for control variables and to ensure uniqueness solution to the (3), the transversality condition is imposed.

In the sequel, I will assume that there is a unique non-trivial stationary solution (called steady-state) to the system $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = F(K^*, C^*)$ and that $\lim_{t\to\infty}[K(t), C(t)] = [K^*, C^*]$. In a typical economic application, this stationary solution exhibits saddle-point stability. Saddle-point stability means that the Jacobian of F evaluated at $[K^*, C^*]$ has so many stable eigenvalues as is the number of linearly independent states and so many unstable eigenvalues as is the number of independent controls. This implies that the optimal path for control variables lies on the stable manifold of the dynamic system. The dimension of this manifold is exactly equal to the number of unstable eigenvalues, which implies that there is a unique optimal path converging to the steady state. Henceforth throughout this paper I will investigate only such problems.

Whenever (3) exhibits the saddle-point stability, the system must not be solved by "shooting" an initial condition for C, since even a small deviation from the equilibrium path (and so also from the initial condition) will cause the system to diverge and to violate the transversality condition. Therefore it is impossible to solve the system (3) as an initial value problem, for which there exists a bulk of efficient methods.

There were proposed different numerical methods for solution of the system (3). See Judd (1998) for a survey. Among them there is a class of methods, which try to convert the hard problem (3) to a much easier initial value problem. These methods started with the seminal contribution of Mulligan and Sala-i-Martin (1991). The authors introduce so-called time-elimination method. This method eliminates the time path on the stable manifold for the control variables by introducing the policy function $C(K)$. Then the system (3) is replaced by the following system:

$$\begin{bmatrix} dK/dt \\ C(t) \end{bmatrix} = \begin{bmatrix} G(K(t), C(K(t))) \\ C(K(t)) \end{bmatrix}, \text{ given } K(0). \tag{4}$$

This system represents indeed an initial value problem and is easy to solve. The policy function $C(K)$ is obtained as a solution to the following differential equation: $C'(K) = \dfrac{G(C,K)}{H(C,K)}$ with an obvious initial condition: $C'(K^*) = 0$.

This method is simple and elegant, however has some drawbacks. The most important is that in some cases the differential equation above is singular and it is not obvious how to continue to integrate the policy function. This occurs especially when the convergence to the steady state is not monotonic.

The backward integration method by Brunner and Strulik (2002) is based on a similar idea, however overcomes the difficulties of the time –elimination method. Their idea is to premultiply the system (3) by –1, change variables ($C(t) = \tilde{C}(-t), K(t) = \tilde{K}(-t)$) and to consider an adjoint dynamical system:

$$\begin{bmatrix} d\tilde{K}/dt \\ d\tilde{C}/dt \end{bmatrix} = \begin{bmatrix} -G(\tilde{K}(t), \tilde{C}(t)) \\ -H(\tilde{K}(t), \tilde{C}(t)) \end{bmatrix} = -F(\tilde{K}(t), \tilde{C}(t)) \tag{5}$$

Note two things: first steady state of the initial system (3) coincides with that of the adjoint system (5) and the unstable manifold of (5) corresponds to the stable manifold of (3). The idea is to approximate the policy function by integrating the unstable manifold from the steady state backward to obtain the policy function. This backward integration represents an initial value problem (easy to solve). Then after obtaining the policy function, we can plug it into the system (4) and solve for the optimal path. The details of the methods are described in Brunner and Strulik (2002).

II. The Environmental Policy Model

To illustrate the backward integration method I introduce a simple growth model with environmental taxation. The exercise performed on this model is too simplistic, but it is used in order to highlight the basic ideas of the backward integration method, described in the preceding section. Anyhow, the model may illustrate basic features of models used for the evaluation of the environmental policy.

The production side of the model is described by the nested CES production function with three inputs: labor, capital and fossil fuels. The production sector consists with a continuum of the unit interval of firms, which means that these firms act as price-takers and do not internalized negative externalities form the fossil fuel usage. The production function has the following form:

$$Y=\left\{\omega_L L^{\rho_1}+\left(\omega_K K^{\rho_2}+\omega_E E^{\rho_2}\right)^{\frac{\rho_1}{\rho_2}}\right\}^{\frac{1}{\rho_1}}$$

where L is used labor, K is capital, E is fossil fuel usage and Y is output. The parameters ω_i, $i\in\{L,K,E\}$ are shares of the inputs and ρ_i $i\in\{1,2\}$are elasticities of substitution among the inputs. I suppose that $\rho_1>\rho_2$, which means that capital and energy are relative substitutes in production. This implies that it is more easily to substitute capital for energy (and vice versa) than labor for the capital-energy bundle. The constant return to scale and the competitiveness assumption implies zero profits in equilibrium. I assume that fossil fuel price is exogenous (given by say the world price) and all other prices (real interest rate and real wage rate) are endogenous to make markets clear.

The fossil fuel usage causes negative environmental externalities X (air pollution emissions), for simplicity I assume the linear dependence: $X=\alpha E$. The measure W how the environment is clean is negatively related to the stock of pollution.

There is a continuum of unit mass of infinitively lived households. They own the firms and are endowed with a unit of time in each date. They value consumption C, leisure – a complement to labor (1-L) - and the clean environment, which is negatively related to the current pollution X as follows:

$$\int_0^{\infty} u(C,1-L,X)e^{-\beta t}dt$$

The time preference parameter satisfies as usually $\beta>0$. While unrealistic, for expositional simplicity I assume that their instantaneous utility functions u are separable in these three arguments. In numerical simulation I assume the following specification for u:

$$u(C,1-L,X)=\frac{C^{1-\sigma}}{1-\sigma}-A\frac{L^{1-\chi}}{1-\chi}-B\frac{X^{1-\gamma}}{1-\gamma}$$

I naturally extend the definition of the function $f(x,\lambda)=x^{1-\lambda}/(1-\lambda)$, which is not defined for $\lambda=1$, in the following way $f(x,1)=\log(x)$.

There is a government, who collect taxes on labor t^L and on fossil fuel usage t^E. Government spending does not affect households or firms. To close the model, the law of motion for the capital is specified: $dK/dt = Y - C - G - \delta K$, which is something standard.

After some simple, but tedious algebra, the model can be converted to the following reduced form:

$$dC/dt = C(r - \beta - \delta)/\sigma \tag{6a}$$

$$dK/dt = rK + wL - C - \delta K \tag{6b}$$

$$L = \left(\frac{AC^{\sigma}}{w}\right)^{\frac{1}{\chi}} \tag{6c}$$

$$r = \frac{\partial Y}{\partial K},\ w = \frac{1}{(1+t^L)}\frac{\partial Y}{\partial L},\ (1+t^E)P_E = \frac{\partial Y}{\partial E} \tag{6d}$$

$$K(0) \text{ given, } \lim_{t\to\infty} e^{-\beta t} K(t)\partial U(C(t), L(t), X(t))/\partial C = 0 \tag{6e}$$

Given endogenous variables C, K and all exogenous variables (t^L, t^E, P_E) and parameters, it is possible to solve for the L, r, w, E, using the equilibrium conditions (6c) and (6d). So, the system (6a-e) is evidently of the form (3). Note that since the agents are atomistic, the agents do not internalized negative externalities. However, the government interventions through taxes may Pareto-improve the outcome.

Especially: increase in taxation of energy with simultaneous decrease in labor income tax leaving government increases unchanged may not only have environmental benefits due to decrease of fossil fuel usage (and therefore the pollution) but also economic benefits in increase of labor demand. The mechanism is that the government revenues from the fossil fuel tax may be used to decrease distortionary taxation of labor. Such hypothesis is called a 'double dividend hypothesis'. This hypothesis is widely discussed, since unemployment is one of the most serious social problems in Europe. See Goulder (1995) for an introduction to the field.

III. Numerical Simulations

Here I will numerically simulate the two scenarios and the transition between them. To simulate the transition path I use the method described in the section I. First scenario is without any 'green' taxes on fossil fuel. The government revenues come only from the taxation of labor. I will compute the steady state and evaluate the utility of the households.

Then I consider the case of introduction of a tax on the fossil fuels with decrease of the labor income tax. I also compute the steady state utility of households. However to assess such 'green' tax reform exactly, it is necessary to compute the transition path between the two steady states. This represents a problem of the form (3) and the backward integration method is performed to do this job.

However before do so, it is necessary to specify numerical values of the parameters. The parameters of the production function are $\delta = 0.15$, $\omega_L = 0.6$, $\omega_K = 0.25$, $\omega_E = 0.15$, $\rho_1 = 0.5$, $\rho_2 = 0.75$. With the value of ρ_i, we can associate the elasticity of substitution between the input factors, see Chung (1994). The parameters for the utility function are set to equal to $\beta = 0.02$, $\sigma = 1$, $\chi = 1$, $A = 0.5$. These values are not inconsistent with econometric findings on the aggregate data in the developed countries.

For the initial scenario I pose $t^L = 0.2$ with $t^E = 0$. As an environmental policy intervention I assume that the policy maker lowers the labor income tax to $t^L = 0.17$ and introduces the environmental tax on fossil fuels in the amount of $t^E = 0.20$.

After computation of the steady states, as expected the fossil fuel usage decrease (about 7%) but the government revenues (from labor income as well as from fossil fuel tax) almost do not change (the decrease is less than 0.07%). However employment increase significantly (about + 16%), the same situation applies to capital stock (it increases about 14%), while wages decrease marginally (about 2%). So, such a green reform not only improve the environment, but also boost the employment and the capital stock (growth), the double dividend is present here. The reason is that even a slight decrease in distortionary taxation of labor income, has a significant positive effect. The effect will not be so strong, if we consider less elastic labor supply. Remember also, that the model assumes a competitive labor market. Effect on a unionized labor market may differ because of trade-off between wages and employment forced by unions.

However this was a description of the steady states corresponding to different policies. But policy-makers are surely concerned with the (speed of) transition between them (think about e.g. political cycles). To simulate this transition between the described steady- states I use the backward integration method. The computed paths for capital, consumption, employment and fossil fuel display the monotonic convergence from the one steady state to another. This is contrary to finding of Oueslati (2002). The reason is that in the presented model, the dynamics of human capital, which can substitute the physical in the long run, was not explicitly modeled. The programs in MATLAB actually used for the numerical simulations are available on a request from the author.

IV. Conclusion

This paper discussed the backward integration method of Brunner and Strulik (2002) for solving the perfect foresight dynamic economic models. The practical examination of this method was demonstrated on computation on the transition path in the simple model of environmental policy. The numerical simulations of the model with realistic parameters suggest the possibility of the so-called environmental 'double-dividend'.

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